(m)-Self-Dual Polygons

THESIS

**Submitted in Partial Fulfillment
of the Requirements for the
Degree of**

BACHELOR OF SCIENCE (Mathematics)

at the

**POLYTECHNIC INSTITUTE OF NEW YORK
UNIVERSITY**
by

Valentin Zakharevich

June 2010

(m)-Self-Dual Polygons

THESIS

Submitted in Partial Fulfillment

of the Requirements for the

Degree of

BACHELOR OF SCIENCE (Mathematics)

at the

POLYTECHNIC INSTITUTE OF NEW YORK UNIVERSITY

by

Valentin Zakharevich

June 2010

Curriculum Vitae:Valentin Zakharevich

Date and Place of Birth:

June 21st 1988, Tashkent, Uzbekistan.

Education:

2006-2010 : B.S. Polytechnic Institute of NYU

January 2009 - May 2009: Independent University of Moscow (Math in Moscow)

September 2008 - December 2008: Penn State University(MASS)

ABSTRACT

(m)-Self-Dual Polygons

by

Valentin Zakharevich

Advisor: Sergei Tabachnikov

Advisor: Franziska Berger

Submitted in Partial Fulfillment of the Requirements

for the Degree of Bachelor of Science (Mathematics)

June 2010

Given a polygon $P = \{A_0, A_1, \ldots, A_{n-1}\}$ with n vertices in the projective space $\mathbb{CP}_2$, we consider the polygon $T_m(P)$ in the dual space $\mathbb{CP}_2^*$ whose vertices correspond to the m-diagonals of P, i.e., the diagonals of the form $[A_i, A_{i+m}]$. This is a generalization of the classical notion of dual polygons where m is taken to be 1. We ask the question, "When is P projectively equivalent to $T_m(P)$?" and characterize all polygons having this self-dual property. Further, we give an explicit construction for all polygons P which are projectively equivalent to $T_m(P)$ and calculate the dimension of the space of such self-dual polygons.

Contents

1 Introduction

The field of projective geometry originated as a study of geometrical objects under projections which was particularly important in fine art and architecture. In order to create an image that has the correct perspective, one needs to know how three-dimensional objects project onto two-dimensional canvas. In modern times, one may find applications of projective geometry in 3D graphics and computer vision.

An important object in this area is the space of lines passing through a given point. Let a point p be fixed in a three dimensional space, and let a plane L be given which does not intersect the point p. We can associate points on the plane L with lines passing through the point p by drawing a line through that point on the plane and p. This covers almost all lines passing through p except for the lines which are parallel to L. For that reason, regular Euclidean planes are not ideal objects to study projections. The ideal object of study is the set of lines passing through the point p. This object, the projective plane, is a plane with additional structure at "infinity". For a more thorough introduction to the field, see [1].

Although the study of projective geometry originates from purely geometric incentives, it possesses a rich algebraic structure. For example, projective spaces are ideal for studying algebraic curves. Also, there are interesting geometric relations in projective spaces that arise from algebraic constructions. In this thesis, we will closely work with the dual projective space which arises from the notion of dual vector spaces.

It turns out that the most natural way to define and study projective spaces is by considering vector spaces. We will assume that the reader is familiar with vector spaces and will not define notions or prove statements that one should learn in a standard Linear Algebra course.

In Section 2 of this thesis we will introduce the notion of projective geometry and projective duality which will serve as the foundation for the work presented in Section 3.

In Section 3 we will present original research that has originated from the work done during an REU(Research Experience for Undergraduates) program at the Penn State University during the summer of 2009. Given an n-gon $P = \{A_0, A_1, \ldots, A_{n-1}\}$ in the projective plane, we will consider the n-gon $T_m(P)$ in the dual projective plane whose vertices correspond to m-diagonals of P i.e., the diagonals of the form $[A_i, A_{i+m}]$. We will construct all n-gons P with the property that P is projectively equivalent to $T_m(P)$ and calculate the dimension of the space of such m-self-dual polygons.

In Section 4 we will present the recently discovered relations between m-self-dual n-gons and those n-gons inscribed into a non-singular conic. These relations are still not very well understood and have partially motivated the research of Section 3.

2 Projective Geometry

2.1 Projective Spaces

Let V be an n-dimensional vector space over a field $\mathbb{k}$. The n-dimensional affine space $\mathbb{A}^n = \mathbb{A}(V)$ is the geometric representation of V. The points of $\mathbb{A}^n$ correspond to vectors of V, and the origin corresponds to the zero vector. The n-dimensional hyperplanes of $\mathbb{A}^n$ are translations of n-dimensional subspaces of V. The definition of affine spaces provides a natural dictionary between the language of algebra and the language of geometry.

Let V be an $(n+1)$-dimensional vector space over a field $\mathbb{k}$. Let $\mathbb{P}_n = \mathbb{P}(V)$ be the projectivization of the vector space V: the points of $\mathbb{P}(V)$ are one-dimensional subspaces of V or, equivalently, lines passing through the origin in $\mathbb{A}(V)$. To visualize the points of the projective space, one uses a screen, i.e., an n-dimensional hyperplane $U \subset \mathbb{A}^{n+1}$ that does not pass through the origin. A point $p \in U$ on the screen corresponds to the line $Op \in \mathbb{P}_n$. The screen is called an affine chart.

Clearly no chart U can cover the entire projective space. The points which are not covered correspond to vectors in U_∞, the translation of U which contains O. Points on U_∞ are called the points at infinity corresponding to the chart U. The space at infinity clearly has the structure of $\mathbb{P}_{n-1} = \mathbb{P}(U_\infty)$, and therefore we can decompose $\mathbb{P}_n = U \cup \mathbb{P}(U_\infty) = \mathbb{A}^n \cup l\mathbb{P}_{n-1}$.

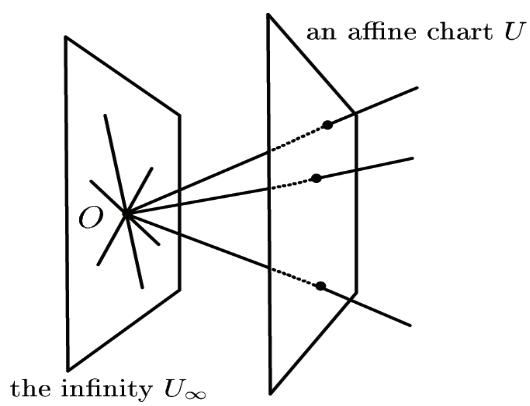

Figure 1: An affine chart

A projective line in projective space $\mathbb{P}(V)$ is the projectivization of a two-dimensional subspace $W \subset V$, which can be written as $\mathbb{P}(W)$. If we take an affine chart U, then $U \cap W$ is either empty (if $W \subset U_\infty$) or consists of a line. Analogously, a k dimensional projective subspace in $\mathbb{P}_n$ is the projectivization of a $k+1$ dimensional subspace of V.

Proposition 2.1. Let U, W be two projective subspaces of $\mathbb{P}_n$ such that $\dim U + \dim W \geq n$, then $U \cap W \neq \emptyset$.

Proof. Let $U', W' \subset V$ be such that $\mathbb{P}(U') = U$ and $\mathbb{P}(W') = W$. Then

$$\dim U' = \dim U + 1,$$
$$\dim W' = \dim W + 1,$$

and

$$\dim U' + \dim W' \geq n + 2.$$

Since $\dim V = n + 1$, $U' \cap W' \neq \{0\}$, or equivalently, $U \cap W \neq \emptyset$ □

In particular, this means that any two lines in $\mathbb{P}_2$ intersect.

2.2 Coordinates

Let V be an $n+1$-dimensional vector space over $\mathbb{k}$ and let $\{e_0, e_1, ..., e_n\}$ be a fixed basis of V. Any two vectors $v = \{v_0, v_1, \ldots, v_n\}$ and $w = \{w_0, w_1, \ldots, w_n\}$, represented in terms of the fixed basis, represent the same point $p \in \mathbb{P}(V)$ if and only if $v_i = \lambda w_i$ for some $\lambda \neq 0$. Therefore only the ratios $\frac{v_i}{v_j}$ are necessary to represent a point p. Thus we write $p = (v_0 : v_1 : \ldots : v_n)$ (defined up to proportionality) for the point represented by the vector $v = \{v_0, v_1, \ldots, v_n\}$.

We would also like to know how to introduce a coordinate system on an affine chart. Let U be an affine chart. There exists a unique $\alpha \in V^*$, where V^* is the vector space dual to V consisting of all linear maps $V \to \mathbb{k}$, such that U is given by the equation $\alpha(x) = 1$ [1]. We will denote this affine chart by U_α. Let $v \in V$ be such that $\alpha(v) \neq 0$. The one dimensional subspace spanned by v intersects U at the point $\frac{v}{\alpha(v)}$. Otherwise, if $\alpha(v) = 0$ then $v \in U_\infty$. To introduce coordinates on the affine chart U_α, fix n linear forms $x_1, x_2, ..., x_n \in V^*$, such that along with α, they form a basis

[1] This is the case since a hyperplane of codimension 1 in $\mathbb{A}^{n+1}$ is given by the equation $\sum_{i=0}^{n} \alpha_i x_i = \beta$. It passes through the origin if and only if $\beta = 0$. Therefore, since an affine chart is a hyperplane that does not pass through the origin, we can write it as $\alpha(x) = \sum_{i=0}^{n} \frac{\alpha_i}{\beta} x_i = 1$

for V^*. To any point $p \in U_\alpha$ which is represented by a vector $v \in V$, we can assign n coordinates $t_i = x_i(\frac{v}{\alpha(v)})$. Clearly these numbers do not depend on the choice of the vector v, since they are the same for λv, where $\lambda \neq 0$. Now given these affine coordinates $(t_1, t_2, \ldots, t_n)$ of a point $p \in U_\alpha$, we can recover the point in the following way. Let e_i be a basis of V such that

$$x_i(e_j) = \delta_{ij}, \qquad i, j = 0, \ldots, n$$

where we take $x_0 = \alpha$. Now it is clear to see that the point p corresponds to the vector $v = e_0 + \sum_{i=1}^{n} t_i e_i$.

Example 2.2. Let $\mathbb{P}_2 = \mathbb{P}(V)$, $\{e_0, e_1, e_2\}$ be a fixed basis of V and $\{x_0, x_1, x_2\}$ the corresponding dual basis such that $x_i(e_j) = \delta_{ij}$ for $i, j = 0, 1, 2$. A polynomial q in the variables x_i does not give a well defined function on $\mathbb{P}_n$, since in general $q(v) \neq q(\lambda v)$, but if q is homogeneous of degree d then $\lambda^d q(v) = q(\lambda v)$, and the set $q(v) = 0$ is a well defined set in $\mathbb{P}_n$. Consider the following homogeneous equation.

$$x_0^2 + x_1^2 = x_2^2$$

This is the equation of a cone in $\mathbb{A}^3$ if we are working over the field $\mathbb{R}$, and we will see how it looks like in different affine charts of $\mathbb{P}$. (Of course we expect to get different conic sections.)

Consider the affine chart U_{x_2}. Let our coordinates on this chart be t_0, t_1 where $t_0 = x_0(\frac{v}{x_2(v)})$ and $t_1 = x_1(\frac{v}{x_2(v)})$. A point p with coordinates (t_0, t_1) corresponds to the vector $t_0 e_0 + t_1 e_1 + e_2$, and equivalently the equation of the cone restricted to U_{x_2} becomes

$$t_0^2 + t_1^2 = 1$$

which is a circle. Analogously if we consider U_{x_0}, $t_1 = x_1|_{U_{x_0}}$ and $t_2 = x_2|_{U_{x_0}}$, then the equation becomes

$$1 + t_1^2 = t_2^2$$

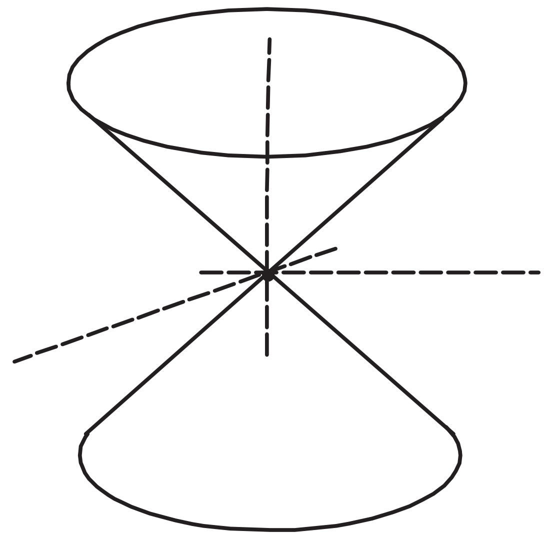

Figure 2: A cone in $\mathbb{R}^3$

which is a hyperbola.

Lastly, consider the chart $U_{x_2-x_1}$ with $t_0 = x_0|_{U_{x_2-x_1}}$ and $t_1 = x_2 + x_1|_{U_{x_2-x_1}}$. Our three linear forms are $x_2 - x_1, x_0, x_0 + x_1$, and the corresponding basis for V is $\{\frac{e_2-e_1}{2}, e_0, \frac{e_2+e_1}{2}\}$. The point p with the coordinates (t_0, t_1) corresponds to the vector

$$t_0e_0 + t_1\frac{e_2+e_1}{2} + \frac{e_2-e_1}{2} = t_0e_0 + \frac{t_1-1}{2}e_1 + \frac{t_1+1}{2}e_2.$$

Therefore, in this chart, our equation takes the following form

$$t_0^2 + \left(\frac{t_1-1}{2}\right)^2 = \left(\frac{t_1+1}{2}\right)^2,$$

which is the parabola

$$t_0^2 = t_1.$$

2.3 Projective Isomorphisms

When studying a particular space, it is natural to consider isomorphisms which preserve important properties of this space. In our case the natural isomorphism is a projective linear isomorphism. Let U, V be two $(n+1)$-dimensional vector spaces. A vector linear isomorphism $f : U \to V$ induces a map $\overline{f} : \mathbb{P}(U) \to \mathbb{P}(V)$ which is called a projective linear isomorphism. $\overline{f}$ takes the point p corresponding to vector v to the point $\overline{f}(p)$ corresponding to the vector $f(v)$. It is clear that the map $\overline{f}$ is well defined, and that two linear isomorphisms f, g from U to V give rise to the same projective linear isomorphism if and only if they are proportional. The map $\overline{f}$ sends k-dimensional hyperplanes to k-dimensional hyperplanes the same way f takes $(k+1)$-dimensional subspaces to $(k+1)$-dimensional subspaces. To justify the term "projective", consider the following example.

Proposition 2.3. Let l_1, l_2 be two lines in $\mathbb{P}_2$, and let $p \notin l_1 \cup l_2$. If the map $\pi_p : l_1 \to l_2$ is the projection that sends a point $q \in l_1$ to the intersection of lines $[q, p] \cap l_2$, then it is a projective linear isomorphism.

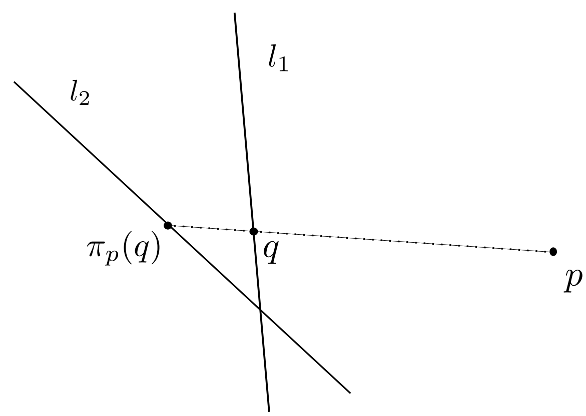

Figure 3: Projection in $\mathbb{P}_2$

Proof. Let $e_0 \in V$ represent $l_1 \cap l_2$. Pick any point $q \in l_1$ not equal to $l_1 \cap l_2$. Let $e_1, e_2 \in V$ be such that e_1 represents q, e_2 represents $\pi_p(q)$, and $e_1 + e_2$ represents p. This can clearly be done since we can scale e_1, e_2 and p lies in the span of $\{e_1, e_2\}$. Now if we let $\{e_0, e_1\}$ be the basis of the subspace corresponding to l_1 and $\{e_0, e_2\}$ be the basis of the subspace corresponding to l_2, the map π_p is induced from the map M, which sends $e_0 \to -e_0$ and $e_1 \to e_2$.[2] To see this, take a point r corresponding to $xe_0 + ye_1$ where $y \neq 0$ and consider the point r' corresponding to $M(xe_0 + ye_1) = -xe_0 + ye_2$. Now it is clear that $\pi_p(r) = r'$ since $p = e_1 + e_2$ lies in the span of $\{xe_0 + ye_1, -xe_0 + ye_2\}$. If r corresponds to e_0, then it is clearly fixed by both π_p and M. □

Definition 1. A set of points $\{p_1, p_2, \ldots, p_m\} \subset \mathbb{P}_n = \mathbb{P}(V)$ is called linearly general if no collection of $(n+1)$ points p_i lies on a hyperplane $\mathbb{P}_{n-1} \subset \mathbb{P}_n$.

Equivalently, this means that for any collection of $(n+1)$ points p_i, the set of vectors representing those points forms a basis for V.

Proposition 2.4. Let $\dim(U) = \dim(V) = (n+1)$ and let $\{p_0, p_1, \ldots, p_{n+1}\} \subset \mathbb{P}(U)$, $\{q_0, q_1, \ldots, q_{n+1}\} \subset \mathbb{P}(V)$ be two linearly general collections of points. Then there exists a unique projective linear isomorphism $M : \mathbb{P}(U) \to \mathbb{P}(V)$ such that $M(p_i) = q_i$ for all i.

Proof. Fix vectors $\overline{p_i} \in U$ and $\overline{q_i} \in V$ such that $\overline{p_i}$ and $\overline{q_i}$ represent points p_i and q_i respectively for all i. Since both collections of points are linearly general, $\{\overline{p_0}, \overline{p_1}, \ldots, \overline{p_n}\}$ and $\{\overline{q_0}, \overline{q_1}, \ldots, \overline{q_n}\}$ form basis for U and V, respectively. If a linear map $\overline{M} : V \to U$ induces a desired map $M : \mathbb{P}(U) \to \mathbb{P}(V)$ then $\overline{M}(\overline{p_i}) = \lambda_i \overline{q_i}$ for some non-zero numbers λ_i and $i = 0, 1, \ldots, n$. Also, since $\{\overline{p_0}, \overline{p_1}, \ldots, \overline{p_n}\}$ and $\{\overline{q_0}, \overline{q_1}, \ldots, \overline{q_n}\}$ are bases for U and V respectively and since the two collections of points are linearly general, there are unique non-zero constants a_i and b_i such that

$$\overline{p_{n+1}} = \sum_{i=0}^{n} a_i \overline{p_i} \qquad \overline{q_{n+1}} = \sum_{i=0}^{n} b_i \overline{q_i}$$

[2]With respect to the given basis, the map M has the following matrix, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and therefore is clearly non-degenerate.

The numbers a_i are not zero for all i since otherwise p_{n+1} would belong to some hyperplane $\mathbb{P}_{n-1} \subset \mathbb{P}_n$ spanned by n points p_i contradicting that the collection $\{p_0, p_1, \ldots, p_{n+1}\}$ is linearly general. Similarly, the numbers b_i are not zero for all i. Also

$$\overline{M}(\overline{p_{n+1}}) = \sum_{i=0}^{n} \lambda_i a_i \overline{q_i}$$

and $M(p_{n+1}) = q_{n+1}$ if and only if $\lambda_i = c\frac{b_i}{a_i}$ for some non zero constant c and all i. These numbers are well defined since $a_i \neq 0$. The map $\overline{M}$ induces the same projective isomorphism M for any value of c which is therefore defined uniquely. □

For example this means that any three points on $\mathbb{P}_1$ can be sent to any other three points on $\mathbb{P}_1$ via a projective linear isomorphism, and there is a unique such map.

2.4 Cross-Ratio

The next natural question to ask is what are some quantities that stay invariant under the action of projective linear isomorphisms. For simplicity, let us focus on $\mathbb{P}_1$. Clearly, the ratio between points is not preserved under projective transformations since any three points can be sent to any other three points. But if we are given four points, then the image of the fourth point under the projective linear transformation which sends the first three to some prescribed points is an invariant. The invariant determined by the image of the fourth point is called the cross-ratio of four points. In this chapter we will derive the expression for the cross-ratio of four points and will show that it is indeed invariant under the action of projective linear isomorphisms.

Let $\mathbb{P}_1 = \mathbb{P}(V)$ and let $\{e_0, e_1\}$ be a fixed basis of V. The set of linear isomorphisms of V can be associated with the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc \neq 0$. Proportional matrices give rise to the same projective linear isomorphisms. The group of non-singular $n \times n$ matrices over a field $\Bbbk$ considered up to proportionality is called projective linear group and is denoted by $PGL_n(\Bbbk)$. We are interested in the action of $PGL_2(\Bbbk)$ on $\mathbb{P}_1$.

The image of a point $(x_0 : x_1)$ under the map corresponding to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $((ax_0 + bx_1) : (cx_0 + dx_1))$. Consider the affine chart U_{x_0} and the affine coordinate $t = x_1|_{U_{x_0}} = \dfrac{x_1}{x_0}$. In this affine chart our map takes the form

$$t \longrightarrow \frac{c + dt}{a + bt}$$

which is a fractional linear transformation considered up to a common multiple.

Consider three affine points on U_{x_0} with coordinates p, q, r. From Proposition 2.4 it follows that there is a unique projective linear isomorphism that will send $p \to 0, q \to 1, r \to \infty$. In this case there is only one point at infinity, namely $(0 : 1)$. To ensure that $p \to 0, r \to \infty$, the map has to be of the form

$$t \longrightarrow \lambda \frac{t - p}{t - r}$$

Solving for λ by evaluating the map at q and putting it back into the expression for the map we get that

$$t \longrightarrow \frac{(q - r)(t - p)}{(q - p)(t - r)}$$

Definition 2. Given four points p, q, r, t on an affine line, we define the cross-ratio of the four points as

$$[p, q, r, t] = \frac{(q - r)(t - p)}{(q - p)(t - r)}$$

Or in other words, $[p, q, r, t]$ is the image of the point t under the projective linear isomorphism that sends $p \to 0, q \to 1, r \to \infty$. This reformulation does not require for the four points to be on any affine chart since it is always possible to find an affine chart which contains any given four points in $\mathbb{P}_1$.

Proposition 2.5. The cross ratio is preserved under the action of $PGL_2(\Bbbk)$ and is independent of the original fixed basis $\{e_0, e_1\}$.

Proof. Let N be a projective linear isomorphism with the property that it sends $p \to 0, q \to 1, r \to \infty$. Consider any projective linear isomorphism M, and the images of p, q, r, t under M. By Proposition 2.4 there exists a unique projective linear isomorphism which sends $M(p) \to 0, M(q) \to 1, M(r) \to \infty$. But NM^{-1} is this map, and therefore $NM^{-1}(M(t)) = N(t)$. It is clear that the cross ratio is independent of the choice of the basis vectors since a change of basis is nothing but an application of a linear isomorphism which we have shown to not effect the cross ratio. $\square$

Example 2.6. We have shown that the projection from a line l_1 to a line l_2 through a point s is a projective linear isomorphism. Right now we deduced that this projection leaves the cross ratio of four points invariant.

$$[p, q, r, t] = [\pi_s(p), \pi_s(q), \pi_s(r), \pi_s(t)]$$

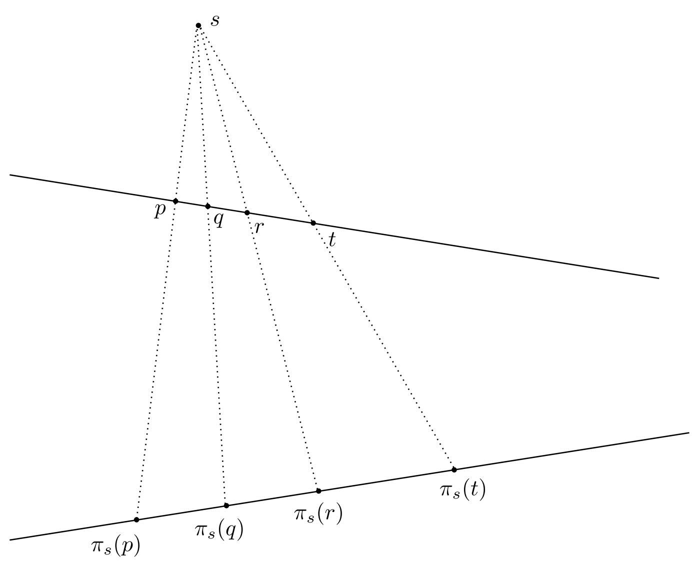

Figure 4: Cross-Ratio under projections

Consider $\mathbb{P}_n$ and a projective isomorphism $f : \mathbb{P}_n \to \mathbb{P}_n$. Given a line $l \in \mathbb{P}_n$, $f|_l$ is a projective isomorphism $l \to f(l)$. Therefore cross-ratio of four collinear points is an invariant under the map f.

2.5 Projective Duality

Let V be a finite-dimensional vector space and let V^* be the vector space dual to V consisting of all linear maps $\alpha : V \to \Bbbk$. If $\alpha \in V^*$ and $v \in V$, we will denote by $\langle \alpha, v \rangle$ the value of α at v: $\alpha(v)$.

Definition 3. Let U be a subspace of V. Let the annihilator of U be $U^\circ \subset V^*$ where

$$U^\circ = \{\alpha \in V^* | \langle \alpha, v \rangle = 0 \text{ for all } v \in U\}.$$

It is an elementary fact from Algebra that $\dim U + \dim U^\circ = \dim V = \dim V^*$. Also, $(U^\circ)^\circ = U$ under the natural identification of V with $(V^*)^*$. This notion of duality becomes very important when we consider the projectivization of V and V^*.

Definition 4. Let V be a vector space, $\mathbb{P} = \mathbb{P}(V)$ and U be a subspace of V. Define the dual projective space as

$$\mathbb{P}^* = \mathbb{P}(V^*).$$

and the dual of $\mathbb{P}(U) \subset \mathbb{P}$ as

$$\mathbb{P}(U)^* = \mathbb{P}(U^\circ) \subset \mathbb{P}^*.$$

Clearly $(\mathbb{P}(U)^*)^* = \mathbb{P}(U)$ for the reason that $(U^\circ)^\circ = U$. The algebraic properties of U° become geometric properties when we consider the duality in the projective space.

Proposition 2.7. Let $H \subset \mathbb{P}_n$ be a hyperplane of dimension k. Then $H^* \subset \mathbb{P}_n^*$ is a hyperplane of dimension $n - k - 1$.

Proof. If H is a hyperplane of dimension k, then $H = \mathbb{P}(U)$, where $U \subset V$ is of dimension $(k+1)$. By definition,

$$H^* = \mathbb{P}(U^\circ)$$

Since $\dim V = (n+1)$ and $\dim U^\circ = (n+1) - (k+1) = (n-k)$,

$$\dim H^* = \dim U^\circ - 1 = n - k - 1$$

□

For the remainder of the section we will deal with properties of duality in $\mathbb{P}_2$. On a projective plane, duality takes lines to points and points to lines. What is important is that projective duality preserves incidences in the following way

a line $l \subset \mathbb{P}_2$	$\longleftrightarrow$	a point $l^* \in \mathbb{P}_2^*$
the points $p \in l$	$\longleftrightarrow$	the lines p^* passing through l^*
the line passing through two points $p_1, p_2 \in \mathbb{P}_2$	$\longleftrightarrow$	the intersection point of lines p_1^*, p_2^*

Proposition 2.8. Let point p be on a line $l \subset \mathbb{P}_2$. Then $l^* \in p^*$.

Proof. Let $\{e_0, e_1, e_2\}$ be a basis of V such that e_0 represents p and the linear span of $\{e_0, e_1\}$ represents l. Also, let $\{x_0, x_1, x_2\}$ be the basis of V^* such that

$$\langle x_i, e_j \rangle = \delta_{ij}$$

It follows that x_2 represents l^* and that the linear span of $\{x_1, x_2\}$ represents p^*. Therefore $l^* \in p^*$. □

Proposition 2.9. Let $p_1, p_2 \in \mathbb{P}_2$, and let l be the line passing through p_1 and p_2. Then p_1^* and p_2^* intersect at l^*.

Proof. By Proposition 2.8, $l^* \in p_1^*$ and $l^* \in p_2^*$, which means exactly that p_1^* and p_2^* intersect at l^*. □

From Example 2.6 we can see that the cross-ratio can be consider as a quantity assigned to four concurrent lines (lines meeting at a single point). Projective duality takes four concurrent lines to four collinear points, and the next proposition shows that the cross-ratio is preserved under duality.

Proposition 2.10. Let $p_1, p_2, p_3, p_4 \in \mathbb{P}_2$ be four points lying on a line L. Let $s \in \mathbb{P}_2$ be a point outside of L. Let l_1, l_2, l_3, l_4 be the lines $[s, p_1], [s, p_2], [s, p_3], [s, p_4]$. Then

$$[p_1, p_2, p_3, p_4] = [l_1^*, l_2^*, l_3^*, l_4^*]$$

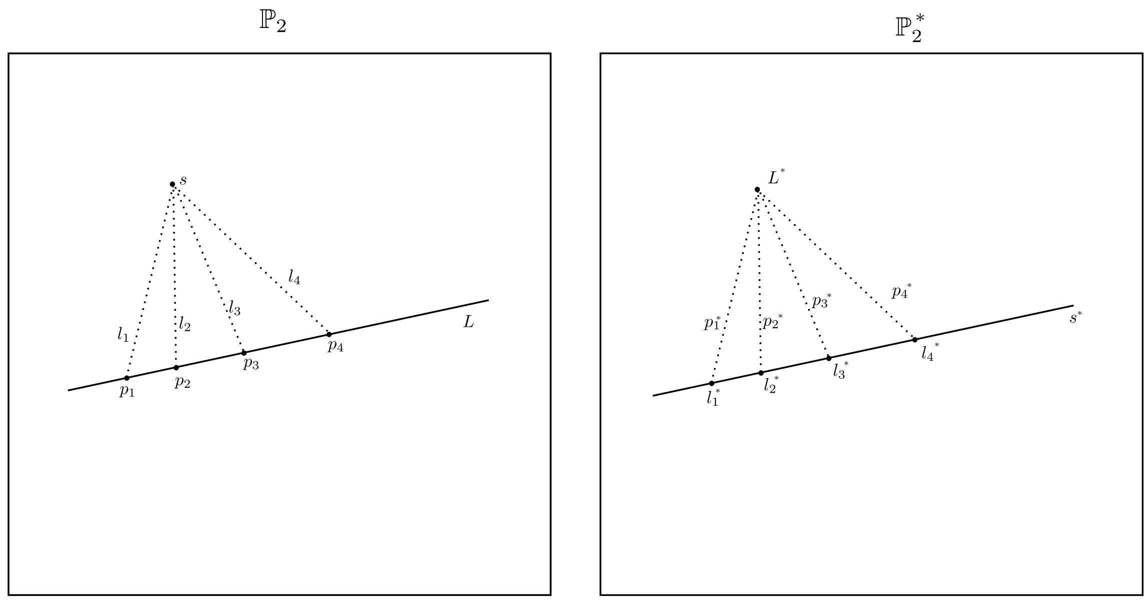

Figure 5: Projective duality and Cross-ratio

Proof. This is done by constructing a projective isomorphism from $\mathbb{P}_2 \to \mathbb{P}_2^*$ that sends the points p_1, p_2, p_3, p_4 to the points $l_1^*, l_2^*, l_3^*, l_4^*$.

Let $\{e_0, e_1, e_2\}$ be a basis for V such that e_0 represents s, and e_1, e_2 represent p_1, p_2 respectively. Since p_1, p_2, p_3, p_4 are distinct points, there exist λ_3, λ_4 such that $e_1 + \lambda_3 e_2$ represents p_3 and $e_1 + \lambda_4 e_2$ represents p_4. Also let $\{x_0, x_1, x_2\}$ be the basis of V^* such that $\langle x_i, e_j \rangle = \delta_{ij}$. It is not hard to see that the lines l_1, l_2, l_3, l_4 and the points $l_1^*, l_2^*, l_3^*, l_4^*$ are represented by the following sets of vectors with respect to this basis.

l_1	$\longleftrightarrow$	$\{e_0, e_1\}$	l_1^*	$\longleftrightarrow$	$\{x_2\}$
l_2	$\longleftrightarrow$	$\{e_0, e_2\}$	l_2^*	$\longleftrightarrow$	$\{x_1\}$
l_3	$\longleftrightarrow$	$\{e_0, e_1 + \lambda_3 e_2\}$	l_3^*	$\longleftrightarrow$	$\{x_2 - \lambda_3 x_1\}$
l_4	$\longleftrightarrow$	$\{e_0, e_1 + \lambda_4 e_2\}$	l_4^*	$\longleftrightarrow$	$\{x_2 - \lambda_4 x_1\}$

Since our original points are given by the following vectors,

$$\begin{array}{ccc} p_1 & \longleftrightarrow & \{e_1\} \\ p_2 & \longleftrightarrow & \{e_2\} \\ p_3 & \longleftrightarrow & \{e_1 + \lambda_3 e_2\} \\ p_4 & \longleftrightarrow & \{e_1 + \lambda_4 e_2\}, \end{array}$$

it is clear that our desired map is the one that sends $e_0 \to x_0, e_1 \to x_2, e_2 \to -x_1$.

□

2.6 Bilinear Forms

Projective duality sends objects in $\mathbb{P}$ to objects in $\mathbb{P}^*$, and therefore, for the duality to have geometrical meaning one should introduce maps from $\mathbb{P}$ to $\mathbb{P}^*$.

Let $\hat{f} : \mathbb{P} \to \mathbb{P}^*$ be a projective linear isomorphism which is induced from the map $f : V \to V^*$. One may define F, a bilinear form on V such that for $u, v \in V$

$$F(u, v) = \langle f(u), v \rangle.$$

Now we can express successive application of duality and the map $\hat{f}^{-1}$ in term of the bilinear form F.

Proposition 2.11. Let U be a subspace of V. Then

$$\hat{f}^{-1}(\mathbb{P}(U)^*) = \mathbb{P}(\{v \in V | F(v, u) = 0 \text{ for all } u \in U\})$$

Proof. The claim follows after rewriting the right hand side of the equality in terms of the map f.

$$\{v \in V | F(v, u) = 0 \text{ for all } u \in U\} = \{v \in V | \langle f(v), u \rangle = 0 \text{ for all } u \in U\} = f^{-1}(U^\circ)$$

□

We can also work in the opposite direction. Given a bilinear form F on V, we can define a map $f : V \to V^*$ as

$$f(v) = F(v, *) \in V^*.$$

F is called non-degenerate if f is an isomorphism.

2.6.1 Symmetric Bilinear Forms

Now we consider a special case when the bilinear form F is symmetric.

Proposition 2.12. Given a non-degenerate symmetric bilinear form F on an $(n+1)$-dimensional vector space V over an algebraically closed field $\Bbbk$, there exists a basis $\{e_0, e_1, \ldots, e_n\}$ of V in which F is given by the identity matrix I, i.e.,

$$F(u, v) = u^t I v$$

Proof. The procedure of finding the desired basis is essentially the same as the Gram-Schmidt method. Pick a vector v_0 such that $F(v_0, v_0) \neq 0$. It is obvious that such vector exists. Let $e_0 = v_0/\sqrt{F(v_0, v_0)}$. This is possible since $\Bbbk$ is algebraically closed. Let

$$U_0 = \{e_0\}^\perp = \{v \in V | F(v, u) = 0 \text{ for all } u \in \{e_0\}\ \}.$$

Any vector v can be decomposed into

$$v = (v - F(v, e_0)e_0) + F(v, e_0)e_0,$$

where $(v - F(v, e_0)e_0)$ is in $\{e_0\}^\perp$ and $F(v, e_0)e_0$ is in $\{e_0\}$. Moreover this decomposition is unique since

$$\{e_0\} \cap \{e_0\}^\perp = \{0\}$$

and we can conclude that $V = \{e_0\} \oplus U_0$. Also, F is non-degenerate on U_0. Repeating the same procedure on U_0 we get e_1 and U_1. After finitely many steps, we end up with a basis $\{e_0, e_1, \ldots, e_n\}$. □

Definition 5. Let F be a fixed symmetric bilinear form on V and $\mathbb{P}(U) \subset \mathbb{P}(V)$. Then

$$\mathbb{P}(U)^\perp = \mathbb{P}(\{v \in V | F(v, u) = 0 \text{ for all } u \in U\})$$

and this duality is called polar duality.

The special name comes from the intimate connection between polar duality with respect to the bilinear form F and the conic defined by the equation

$$F(v, v) = 0.$$

Lets investigate what polar duality looks like. Let a basis of V in which the bilinear form F is given by the identity matrix I be fixed. Then for a given point $(v_0 : v_1 : \ldots : v_n) \in \mathbb{P}(V)$ we have

$$(v_0 : v_1 : \ldots : v_n)^\perp = \{(x_0 : x_1 : \ldots : x_n) \in \mathbb{P}(V) | \sum_{i=0}^{n} x_i v_i = 0\}$$

The following remark tells us how much freedom we have while choosing the basis in Proposition 2.12.

Definition 6. Let $O_n(\mathbb{k})$ be the set of orthogonal $n \times n$ matrices A with entries in $\mathbb{k}$, such that

$$AA^t = I.$$

Remark. Let V be a vector space over a field $\mathbb{k}$ with a fixed basis and F be a bilinear form given by the identity matrix I with respect to this basis. It is a well known fact that the set of isomorphisms f of V which leave the bilinear form F invariant, i.e., $F(u,v) = F(f(u), f(v))$, is given by the set $O_{n+1}(\mathbb{k})$ if we associate linear maps $V \to V$ with $(n+1) \times (n+1)$ matrices via the given basis.

The next theorem will be important for us in the next section. It implies that up to the action of $O_3(\mathbb{k})$, there are only three pairs of points $p, q \in \mathbb{P}_2$ such that $q \in p^{\perp}$.

Theorem 2.13. Let V be a 3-dimensional vector space over an algebraically closed field $\mathbb{k}$ and let F be a non-degenerate symmetric bilinear form on V. Let $v_1, v_2 \in V$ and $u_1, u_2 \in V$ be two collections of linearly independent vectors such that

$$\begin{aligned} F(v_1, v_2) &= F(u_1, u_2) = 0 \\ F(v_1, v_1) &= F(u_1, u_1) = a_1 \\ F(v_2, v_2) &= F(u_2, u_2) = a_2 \end{aligned}$$

and the a_i are equal to 0 or 1. Then at most one $a_i = 0$ and there exists a unique map $g \in O_3(\mathbb{k})$ such that $g(v_i) = u_i$.

Proof. First we show that both a_i are not 0. Assume they are. We will show that the map $f : V \to V^*$ given by

$$f(v) = F(v, *)$$

is not an isomorphism which will in turn imply that F is degenerate, producing a contradiction. Fix a vector v_3 such that $\{v_1, v_2, v_3\}$ forms a basis of V and let $\{x_1, x_2, x_3\}$ be the associated dual basis. The fact that

$$F(v_1, v_2) = 0 \qquad\qquad F(v_1, v_1) = 0$$

implies that $f(v_1)$ is in the span of x_3. By the same logic, $f(v_2)$ is also in the span of x_3. This means that f is not an isomorphism.

Now there are two cases to consider:

Case 1: Assume $a_1 = a_2 = 1$. Then there exist unique v_3, u_3 such that in both bases $\{v_1, v_2, v_3\}$ and $\{u_1.u_2, u_3\}$ F is given by the identity matrix I. This can be seen from the proof of Proposition 2.12. In this case it is obvious that the only orthogonal map with the desired conditions is the one that sends $g(v_i) = u_i$.

Case 2: Without loss of generality assume that $a_1 = 0$ and $a_2 = 1$. The outline of the argument is the following. We will show that there are unique vectors v_3 and u_3 such that

$$\begin{aligned} F(v_3, v_1) &= F(u_3, u_1) = 1 \\ F(v_3, v_2) &= F(u_3, u_2) = 0 \\ F(v_3, v_3) &= F(u_3, u_3) = 0 \end{aligned}$$

and that the sets $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, u_3\}$ form bases for V. In this case the only map satisfying the desired properties is again the one that sends $g(v_i) = u_i$. The reason for that is that if g is orthogonal, then $g(v_3)$ must possess all the properties that u_3 has. Since we will show that u_3 is the only vector with such properties, we will have shown that $g(v_3)$ must be u_3.

We will show the existence and uniqueness of v_3. The argument for u_3 is the same. Since

$$F(v_2, v_1) = 0 \qquad\qquad F(v_2, v_2) = 1,$$

we know that $v_2 \notin \{v_2\}^{\perp}$.Therefore for $\{v_1, v_2, v_3\}$ to form a basis and for $F(v_3, v_2) = 0$, v_3 must be of the following form

$$v_3 = \lambda_1 v_1 + \lambda_2 v_3'$$

where $\lambda_2 \neq 0$ and v_3' is some vector such that $v_3' \in \{v_2\}^{\perp}$. Since $F(v_3, v_1) = 1$ we get that

$$F(v_3, v_1) = \lambda_1 F(v_1, v_1) + \lambda_2 F(v_1, v_3') = \lambda_2 F(v_1, v_3') = 1$$

or equivalently

$$\lambda_2 = \frac{1}{F(v_1, v_3')}$$

which is defined and is unique since $v_3' \notin v_1^\perp = \{v_1, v_2\}$. Also since $F(v_3, v_3) = 0$ we get that

$$\begin{aligned} F(v_3, v_3) &= \lambda_1^2 F(v_1, v_1) + 2\lambda_1\lambda_2 F(v_1, v_3') + \lambda_2^2 F(v_3', v_3') \\ &= 2\lambda_1 + \frac{F(v_3', v_3')}{F(v_1, v_3')^2} = 0 \end{aligned}$$

Which fixes λ_1 uniquely.

□

2.6.2 Non-symmetric Bilinear Forms

In this section we investigate how many "distinct" non-symmetric forms there are. We will focus on a three-dimensional vector space V over the field $\mathbb{C}$ since that is what we need for the next section. The next lemma will be important for us in order to analyze non-symmetric bilinear forms.

Definition 7. A bilinear form F on V is called skew-symmetric if $F(u, v) = -F(v, u)$ for all $u, v \in V$.

Lemma 2.1. Let F be a bilinear form on V. Then there exist a symmetric bilinear form F_+ and a skew-symmetric bilinear form F_- such that

$$F = F_- + F_-.$$

Further, these bilinear forms are unique.

Proof. Fix a basis for V so that F can be associated with a matrix that we will again call F. Then let

$$F_- = \frac{F - F^t}{2} \qquad\qquad F_+ = \frac{F + F^t}{2}.$$

It is clear that $F = F_- + F_+$. The facts that F_- is skew-symmetric and that F_+ is symmetric follow from the following relation

$$u^t F v = v^t F^t u.$$

Assume that there are distinct matrices F'_- and F'_+ which have those properties. Then

$$F'_- + F'_+ = F_- + F_+$$
$$F_- - F'_- = F_+ - F'_+.$$

Since both sides of the last equality must be symmetric and skew-symmetric, we know that they must be exactly 0. Therefore $F'_- = F_-$ and $F'_+ = F_+$. $\square$

The following is a classical lemma concerning skew-symmetric bilinear forms

Lemma 2.2. Let F be a skew-symmetric bilinear form on V. Let $f : V \to V^*$ be defined as

$$f(v) = F(v, *).$$

Then rank of f is even.

Proof. Let $U \subset V$ such that $f|_U : U \to f(U)$ is an isomorphism. Our goal is to prove that $\dim U = n$ is even. The first claim that we want to make is that we can consider the map $f|_U$ as a map from U to U^*. The space V can be decomposed into

$$V = U \oplus \ker f,$$

and therefore V^* can be decomposed into

$$V^* = U^* \oplus (\ker f)^*$$

where $U^* = (\ker f)^\circ$ and $\ker f^* = U^\circ$. We need to show that $f(U) \subset U^*$. Assume it is not. Then there exists $u \in U$ such that $f(u) \notin (\ker f)^\circ$ or equivalently that there exists $v \in \ker f$ such that $\langle f(u), v \rangle \neq 0$. But this is a contradiction since

$$\langle f(u), v \rangle = F(u, v) = -F(v, u) = -\langle f(v), u \rangle = 0.$$

By abuse of notation we write f for $f|_U$. Pick any vector e_1 in U. Consider the following set,

$$e_1^{\perp} = \{v \in U | \langle f(e_1), v \rangle = 0\}$$

Since $e_1^{\perp} = f(e_1)^{\circ}$ we know that $\dim e_1^{\perp} = n - 1$. Also $e_1 \in e_1^{\perp}$ since $F(e_1, e_1) = -F(e_1, e_1) = 0$. This means that there exists a vector f_1, not a multiple of e_1 such that $U = e_1^{\perp} \oplus \{f_1\}$. In particular $\langle f(e_1), f_1 \rangle \neq 0$. By scaling f_1, we can assume that

$$\langle f(e_1), f_1 \rangle = 1.$$

Let $W_1 = \{e_1, f_1\}$ and let

$$U_1 = W_1^{\perp} = \{u \in U | \langle v, u \rangle = 0 \quad \text{for all } v \in W_1\}$$

Since $U_1 = f(e_1)^{\circ} \cap f(f_1)^{\circ}$, we can conclude that $\dim U_1 = n - 2$ since it is an intersection of two distinct $(n-1)$-dimensional subspaces. Also $U_1 \cap W_1 = \{0\}$ since neither e_1 or f_1 are in U_1 (since $F(e_1, f_1) = 1$). This means that

$$U = U_1 \oplus W_1.$$

Also, since $F(u, v) = 0$ for all $u \in U_1$, $v \in W_1$, we can consider the map

$$f|_{U_1} : U_1 \to U_1^*.$$

Repeating the same procedure for U_1 we get vectors $\{e_2, f_2\}$. After finitely many steps we will have found basis for U which consists of $\{e_1, e_2, \ldots, e_{\frac{n}{2}}, f_1, f_2, \ldots, f_{\frac{n}{2}}\}$. This basis is called symplectic basis of U. It is clear that $\dim U$ must be even. □

Theorem 2.14. Let F be a non-degenerate non-symmetric bilinear form on a vector space $V = \mathbb{C}^3$. Then there exists a basis in V with respect to which F has one of the following forms:

$$H_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad J = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad K = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad .$$

Also, for the case H_ϕ if 2ϕ is a multiple of π then the dimension of the group of transformations preserving H_ϕ is 3, and otherwise it is 1.

Proof. The first thing that is important to see is that, given a basis $\{e_1, e_2, e_3\}$ of V, F is given by the matrix $M = (a_{ij})$, where $a_{ij} = F(e_i, e_j)$.

Let F_-, F_+ be skew-symmetric and symmetric bilinear forms respectively such that $F = F_- + F_+$. Let $f_-, f_+ : V \to V^*$ be the associated maps. We know that $F_- \neq 0$ since F is non-symmetric. Since by Lemma 2.2 the rank of f_- is even, and we know it is not 0, it must be 2. Let $W = \ker f_-$. Since rank of f_- is 2, $\dim W = 1$.

Case 1: $F_+|_W \neq 0, \operatorname{rank} f_+ = 3$.

Fix $e_3 \in W$ such that $F_+(e_3, e_3) = 1$. This is possible since $F_+|_W \neq 0$. Also since W is one-dimensional, e_3 is unique. Let $Z = W^\perp$ with respect to F_+, i.e.,

$$Z = \{v \in V | F_+(u, v) = 0 \quad \text{for all } u \in W\}.$$

Fix $e_1', e_2' \in Z$ such that $F_+(e_i', e_j') = \delta_{ij}$. This is possible by Proposition 2.12 since F_+ is a non-degenerate bilinear form on Z. There is one degree of freedom while choosing e_1', e_2'. This is the case since one only needs to fix the direction of e_1' (the relation $F_+(e_1', e_1') = 1$ defines the scaling). Therefore, e_1' could be thought of as being chosen on $\mathbb{P}(Z)$ which is one dimensional. Also e_2' is fixed by the choice of e_1' and therefore does not add to the number of degrees of freedom.

Now let $e_1 = \lambda e_1'$ and $e_2 = \lambda e_2'$. We will find the value of λ such that F is given by

H_ϕ in the basis $\{e_1, e_2, e_3\}$. In this basis F is given by

$$M = \begin{pmatrix} \lambda^2 & \lambda^2 F_-(e_1', e_2') & 0 \\ -\lambda^2 F_-(e_1', e_2') & \lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

There are four values of λ such that

$$\det M = \lambda^4 + \lambda^4 F_-(e_1', e_2')^2 = 1,$$

since $\det M \neq 0$ (F is non-degenerate). Picking any of those values of λ, we get that $M = H_\phi$ where ϕ is given by $\cos\phi = \lambda^2$. Also we know that $\lambda \neq 0$, therefore $\phi \neq k\pi/2$. The choice of λ does not add to the number of degrees of freedom for the choice of basis since there are finitely many choices for λ.

Case 2: $F_+|_W \neq 0, \operatorname{rank} f_+ = 2$.

Let e_3 and Z be the same as in Case 1. Let $f_+ : V \to V^*$ be the map associated with F_+. Since $F_+|_W \neq 0$, $\ker f_+ \subset Z$. Also, Z is two-dimensional and $\ker f_+$ is one dimensional, therefore we can pick $e_1 \in Z - \ker f_+$ such that $F_+(e_1, e_1) = 1$. This is possible since if $v \in Z - \ker f_+$, then $F_+(v, v) \neq 0$, since otherwise v would be in $\ker f_+$. Also fix some $e_2' \in \ker f_+$. Let $e_2 = \lambda e_2'$. For an appropriate value of λ, in the basis $\{e_1, e_2, e_3\}$, F is given by the matrix J. In this basis F is given by

$$M = \begin{pmatrix} 1 & \lambda F_-(e_1, e_2') & 0 \\ -\lambda F_-(e_1, e_2') & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore choosing

$$\lambda = \frac{1}{F_-(e_1, e_2')}$$

gives us the desired basis.

Case 3: $F_+|_W \neq 0, \operatorname{rank} f_+ = 1$.

In this case, fix e_3 in the same way as in Case 1. Let f_+ be the same as in Case 2. Pick $e_1, e_2 \in \ker f_+$ such that $F_-(e_1, e_2) = 1$. This is obviously possibly since F_- is non-degenerate on $\ker f_+$. In the basis $\{e_1, e_2, e_3\}$, F is given by the matrix

$$H_{\frac{\pi}{2}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now the question is how many degrees of freedom we have while choosing the symplectic basis for the space $\ker f_+$. Clearly e_1 can be chosen anywhere in $\ker f_+$ which adds 2 degrees of freedom. Once e_1 is chosen, to choose e_2, we have to pick a point on $\mathbb{P}(\ker f_+)$ (the scaling is fixed by the relation $F_-(e_1, e_2) = 1$). Therefore, there are 3 degrees of freedom while choosing the basis in this case.

Case:4 $F_+|_W = 0, \operatorname{rank} f_+ = 3$.

Let Z be the same as in the cases above. Fix some $e_3' \in W$. Consider the set $C = \{v \in V | F_+(v, v) = 0\}$. In a basis in which F_+ is given by the matrix I, this set is given by the equation

$$x_1^2 + x_2^2 + x_3^2 = 0.$$

Therefore it does not belong to any two dimensional subspace. In particular $C \not\subset Z$. Let $e_2' \in C - Z$. Then $F_+(e_2', e_2') = 0$ and $F_+(e_2', e_3') \neq 0$. By scaling e_2', we may assume that $F_+(e_2', e_3') = 1$. Take $e_1 \in \{e_3'\}^\perp \cap \{e_2'\}^\perp$, where the orthogonal compliment is taken with respect to F_+.The set $\{e_1, e_2', e_3'\}$ forms a basis. This is true because if e_1 was a combination of e_2' and e_3', it could not be perpendicular to both e_2 and e_3 since

$$F_+(e_2', e_2') = 0 \qquad F_+(e_3', e_3') = 0 \qquad F_+(e_2', e_3') = 1$$

For this reason we have that $F_+(e_1, e_1) \neq 0$, because otherwise e_1 would be perpendicular to all three basis vectors contradicting the non-degeneracy of f_+. Therefore we can scale e_1 so that $F_+(e_1, e_1) = 1$. Let $e_2 = \lambda e_2'$ and $e_3 = \lambda^{-1} e_3'$. For an appropriate value of λ, in the basis $\{e_1, e_2, e_3\}$, F is given by the matrix K. In this basis F is given by

$$M = \begin{pmatrix} 1 & \lambda F_-(e_1, e_2') & 0 \\ -\lambda F_-(e_1, e_2') & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Also $F_-(e_1, e_2') \neq 0$ because otherwise F_- would be 0. Therefore setting

$$\lambda = \frac{1}{F_-(e_1, e_2')}$$

gives us the desired basis.

Case 5: $F_+|_W = 0, \operatorname{rank} f_+ < 3$.

Let $f : V \to V^*$ be the map associated with F. Let U be a one dimensional subspace of V such that $U \subset \ker f_+$. If $U = W$, then F is degenerate since for any $u \in U$, $f(u) = 0$. Therefore assume that $U \neq W$. Let $e_1 \in U, e_2 \in W, e_3 \in V$ be such that $\{e_1, e_2, e_3\}$ is a basis of V. Also let $\{x_1, x_2, x_3\}$ be the basis of V^* such that

$$\langle x_i, e_j \rangle = \delta_{ij}.$$

Then both $f(e_1)$ and $f(e_2)$ are in the span of $\{x_3\}$ since

$$F(e_1, e_1) = F(e_1, e_2) = F(e_2, e_2) = 0.$$

Also, $F(e_2, e_2) = 0$ since $F_+|_W = 0$. This contradicts the non-degeneracy of F.

The dimension of the group of transformations preserving the basis is equal to number of degrees of freedom in the choice of the basis. Looking back at Case 1: and Case 3:, we see that for the case H_ϕ, if 2ϕ is a multiple of π then the dimension of the group of transformations preserving H_ϕ is 3, and otherwise it is 1.

$\square$

3 (l, m)-Self-Dual Polygons

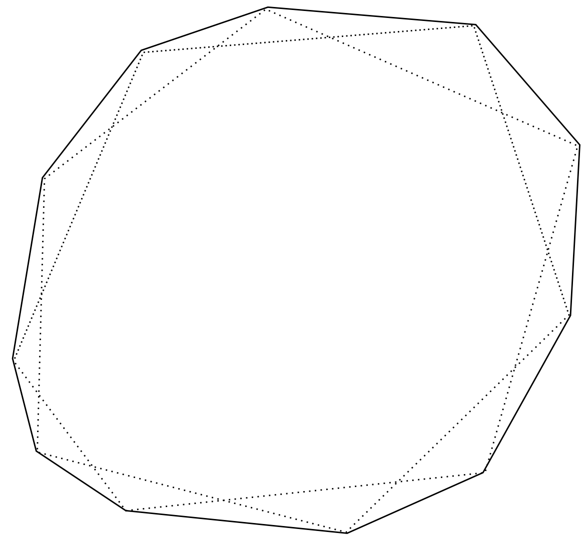

Figure 6: 2-Diagonals

The classical notion of duality for polygons uses the edges of a polygon to construct the vertices of the dual polygon. In particular, if one considers polygons in $\mathbb{P}_2$ over $\mathbb{R}$ or $\mathbb{C}$, one can get the dual polygon by considering duals of the edges of the original polygon. In [2], Dmitry Fuchs and Serge Tabachnikov have studied n-gons for which the dual polygon is projectively equivalent to the original polygon up to cyclic permutation of the vertices. In their paper, they derived the dimension of the set of such self-dual polygons and also presented a way of constructing all such polygons.

In this thesis we generalize the results of the above paper to a more general notion of duality for polygons. In particular, given an n-gon $P = \{A_0, A_1, \ldots, A_{n-1}\}$, we will consider the m-diagonals of P, i.e., the diagonals of the form $[A_i, A_{i+m}]$ which form an n-gon in the dual space. We will classify all n-gons in $\mathbb{CP}_2$ which are projectively equivalent to their dual n-gons in this general sense as well as deduce the dimension of the set of such self-dual polygons.

In this section we will talk about dimensions of spaces of polygons. There are different ways one can define dimension on a space but most of them coincide in a proper context. In this thesis we will not define formally what we mean by the word dimension. Instead, a space will be called n-dimensional if it has n "degrees of freedom" or equivalently, locally, it can be given by n coordinates.

3.1 Notation

We are going to consider n-gons $P = \{A_0, A_1, \ldots, A_{n-1}\}$ where $A_i \in \mathbb{P}(\mathbb{C}^3)$ and indices are considered modulo n, for which the vertices A_i, A_{i+m}, A_{i+2m} are not collinear for all i for a fixed m. In particular this implies that $2m \neq n$. Let $B_i^m = [A_i A_{i+m}]$ and let $B_i^{m*} \in \mathbb{P}^*$ be dual to B_i^m.

Definition 8. Let P be an n-gon. We are going to say that P is an (l, m) self-dual n-gon if there exists a projective isomorphism $\hat{f} : \mathbb{P} \to \mathbb{P}^*$ such that $\hat{f}(A_i) = {B_{i+l}^m}^*$.

We will require that $0 \leq l < n$, $0 < m < n$, $l + m < n$ and $2l + m \leq n$. The third inequality is not a restriction since if P is (l, m) self dual such that $l + m \geq n$, then P is also (l', m') self-dual where $l' = l + m \mod n$ and $m' = n - m$. The last inequality is also not a restriction since if $2l + m > n$, we can orient the polygon in the other direction making it an (l', m) self-dual polygon where $l' = n - (l + m)$ and $2l' + m = 2n - (2l + m) < n$.

Given an (l, m) self-dual n-gon with an associated projective isomorphism $\hat{f}$, we can fix an isomorphism $f : \mathbb{C}^3 \to \mathbb{C}^{3*}$ that induces $\hat{f}$, and a bilinear form $F : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}$ such that $F(v, u) = \langle f(v), u \rangle$. While $\hat{f}$ is unique, f and F are unique up to multiplication by a non-zero constant. By abusing notation we are going to let A_i mean both, a one dimensional subspace of $\mathbb{C}^3$ and a non-zero vector in this subspace.

Definition 9. Let P be an n-gon $P = \{A_0, A_1, \ldots, A_{n-1}\}$. Then $kP = \{A_0, \ldots, A_{kn-1}\}$ is a $(kn) - gon$ where $A_i = A_{i+rn}$ for all $1 \leq r < k$. A polygon P will be called simple if $P \neq kP'$ for any k and any polygon P'.

We will consider only simple polygons. This is justified by the fact that if P is (l, m) self-dual, then so is kP.

3.2 The Case $n = 2l + m$

Theorem 3.1. Let P be an (l, m) self-dual n-gon. Then the bilinear form F is symmetric if and only if $n = 2l + m$.

Proof. First notice that

$$\begin{aligned} F(A_i, A_j) = 0 &\iff \langle f(A_i), A_j\rangle = 0 \iff \langle {B^m_{i+l}}^*, A_j\rangle = 0 \\ &\iff A_j \in B^m_{i+l} = [A_{i+l}A_{i+l+m}]. \end{aligned}$$

In particular, $F(A_i, A_{i+l}) = F(A_i, A_{i+l+m}) = 0$. Let F be symmetric. Then

$$\begin{aligned} F(A_{i+l}, A_i) = 0 &\iff A_i \in B^m_{i+2l} \\ F(A_{i+l+m}, A_i) = 0 &\iff A_i \in B^m_{i+2l+m} \end{aligned} \tag{3.2.1}$$

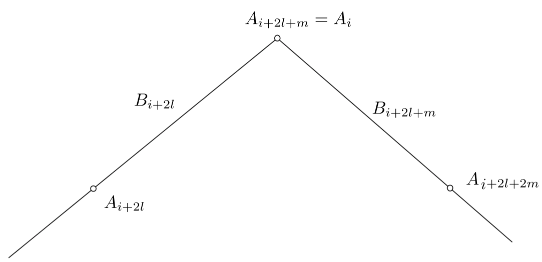

Figure 7: The case when F is symmetric.

Therefore, if F is symmetric, then $A_i = B^m_{i+2l} \cap B^m_{i+2l+m} = A_{i+2l+m}$. Since $2l+m \le n$, the polygon is simple and this is true for all i, it follows that $n = 2l+m$.

Now assume that $n = 2l + m$. From equation (3.2.1), we can conclude that $F(A_{i+l}, A_i) = F(A_{i+l+m}, A_i) = 0$. Since the one-forms $F(*, A_i)$ and $F(A_i, *)$ are non-zero and have the same kernel, i.e., the linear span of $\{A_{i+l}, A_{i+l+m}\}$, they are proportional. Let $\lambda_i \neq 0$ be such that $F(A_i, *) = \lambda_i F(*, A_i)$. In order to demonstrate that F is symmetric, we need to show that $\lambda_i = \lambda_j = 1$ for some $i \neq j$. In the case when such i, j exist, we can pick any vector e such that $\{A_i, A_j, e\}$ forms a basis for the vector space $\mathbb{C}^3$. Then for any $x, y \in \mathbb{C}^3$, if we expand $F(x, y)$ in this basis, every mixed term will contain either A_i or A_j and therefore can be reversed, proving that F is symmetric.

Clearly if $F(A_i, A_i) \neq 0$ then $\lambda_i = 1$. We will show that there exist i, j such that $F(A_i, A_i) \neq 0$ and $F(A_j, A_j) \neq 0$ which implies that $\lambda_i = \lambda_j = 1$ and consequently that F is symmetric.

We will argue by contradiction. Assume that $F(A_i, A_i) = 0$ for at least $(n-1)$ values of i. Fix a value for i such that $F(A_i, A_i) = 0$. We will show that $F(A_{i+l}, A_{i+l}) \neq 0$ and $F(A_{i+l+m}, A_{i+l+m}) \neq 0$ which will contradict the assumption.

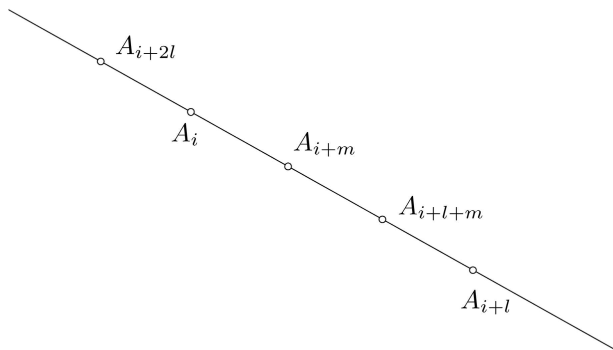

Figure 8: Assuming $F(A_i, A_i) = F(A_{i+l}, A_{i+l}) = 0$.

First notice that $A_i \in B^m_{i+l} = [A_{i+l}A_{i+l+m}]$ since $F(A_i, A_i) = 0$. Assume that $F(A_{i+l}, A_{i+l}) = 0$ then also $F(A_{i+l+m}, A_{i+l+m}) = 0$ since

$$F(A_i, A_i) = F(A_{i+l}, A_{i+l}) = F(A_{i+l}, A_i) = F(A_i, A_{i+l}) = 0$$

and A_{i+l+m} is in the span of $\{A_i, A_{i+l}\}$ since these three points are collinear. Now applying the same argument to points A_{i+l} and A_{i+l+m} we get

$$\begin{aligned} A_{i+l} &\in B^m_{i+2l} = [A_{i+2l}, A_{i+2l+m}] = [A_{i+2l}, A_i] \\ A_{i+l+m} &\in B^m_{i+2l+m} = [A_{i+2l+m}, A_{i+2l+2m}] = [A_i, A_{i+m}]. \end{aligned}$$

Thus, the points $A_{i+2l}, A_i, A_{i+m}, A_{i+l+m}, A_{i+l}$ lie on a line. This is impossible since A_{i+2l}, A_i, A_{i+m} cannot be collinear due to the non-degeneracy of the polygon P. Therefore $F(A_{i+l}, A_{i+l}) \neq 0$.

The fact that $F(A_{i+l+m}, A_{i+l+m}) \neq 0$ then follows from the same line of reasoning: if $F(A_{i+l+m}, A_{i+l+m}) = 0$, then $F(A_{i+l}, A_{i+l}) = 0$ since

$$F(A_i, A_i) = F(A_{i+l+m}, A_{i+l+m}) = F(A_{i+l+m}, A_i) = F(A_i, A_{i+l+m}) = 0,$$

and since A_{i+l} is in the span of $\{A_i, A_{i+l+m}\}$.

□

3.2.1 Explicit Construction

Here we are going to explicitly construct all (l, m) self-dual n-gons where $n = 2l + m$. Since we know that the bilinear form F is symmetric, in appropriate coordinates the duality becomes polar duality.

$$(a : b : c) \to (ax + by + cz = 0)$$

If we consider the chart U_{x_2}, then to get a polar line of a point one reflects the point in the unit circle, then reflects it in the origin and then draws a line through the given point perpendicular to its position vector. This can easily be seen from the fact that in U_{x_2}, polar duality has the following form:

$$(a, b) \to (ax + by + 1 = 0).$$

Once we fix this polar duality, we consider polygons up to the action of $O_3(\mathbb{C})$, which preserves polar duality.

Theorem 3.2. A polygon $P = \{A_0, A_1, ...A_{n-1}\}$ is (l, m) self dual with respect to polar duality where $n = 2l + m$ if and only if $A_{i+l} \in A_i^{\perp}$ for all i.

Proof. One direction is obvious. If P is an (l, m) self-dual n-gon, with respect to polar duality, then $A_{i+l} \in A_i^{\perp} = [A_{i+l}, A_{i+l+m}]$ by definition. For the other direction assume that $A_{i+l} \in A_i^\perp$ for all i. We need to show that $A_{i+l+m} \in A_i^\perp$ for all i. But

$$A_i = A_{i+2l+m} \in A_{i+l+m}^\perp \Longrightarrow A_{i+l+m} \in A_i^\perp.$$

The implication follows from the fact that the form F is symmetric. □

Now we can construct all (l, m) self-dual n-gons where $n = 2l + m$. Pick A_0 and A_l such that $A_l \in A_0^\perp$. By Theorem 2.13, there are three choices for A_0, A_l modulo the action of $O_3(\mathbb{C})$. Then continue choosing $A_{kl} \in A_{(k-1)l}^\perp$ and $A_{kl} \neq A_{(k-2)l}$ until you get to A_{-l} and choose $A_{-l} = A_{-2l}^\perp \bigcap A_0^\perp$. Also, when choosing A_{-2l}, one should ensure that $A_{-2l} \neq A_0$.

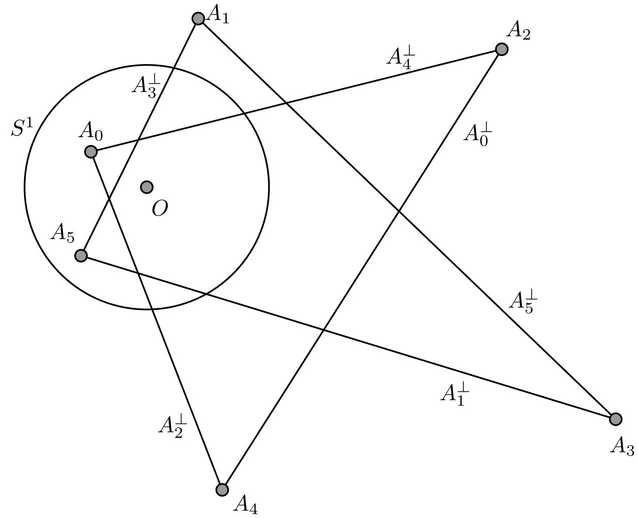

Figure 9: Construction of a $(2, 2)$ self-dual 6-gon with respect to polar duality.

If you exhausted all vertices, then you are done. In this case the dimension of the moduli space of all (l, m) self-dual n-gons is $n - 3$. All vertices except A_0, A_l, A_{-l} were chosen on a line with a finite number of points removed. If, on the other hand, $(l, n) \neq 1$ where (l, n) denotes the greatest common divisor of l and n, then take A_1 anywhere in the plane, and perform the same procedure starting from A_1. Repeat the same procedure starting from A_2 and so on, until you generate all the points.

In this case the moduli space of (l, m) self-dual n-gons is again $n-3$, since when you perform this procedure starting from A_1, you are picking A_1 anywhere in the plane, and the subsequent points on lines except for the last one which is fixed by previous choices which gives you the dimension $\frac{n}{(n,l)}$. Therefore, the total dimension is $\frac{n}{(l,n)} - 3 + ((l,n)-1)\frac{n}{(l,n)} = n-3$.

3.3 The Case $n \neq 2l + m$

Let $P = A_0, A_1, \ldots, A_{n-1}$ be an (l, m) self-dual n-gon where $n \neq 2l + m$. From Theorem 3.1 we know that the bilinear form F is non-symmetric.

Definition 10. Let B be a line or a point in the space $\mathbb{P}$. Define

$$B^{\perp} = \{y \in \mathbb{P} | F(x, y) = 0 \text{ for all } x \in B\},$$

and let $G : \mathbb{P} \to \mathbb{P}$ be defined as $G(B) = (B^{\perp})^{\perp}$.

In matrix form, $G = F^{-1}F^t$. This is the case since $x^{\perp} = \ker(x^t F)$ and

$$(x^{\perp})^{\perp} = \{z | y^t F z = z^t F^t y = 0 \text{ for all } y \in x^{\perp} = \ker(x^t F)\}.$$

Therefore, up to a constant, $z^t F^t = x^t F$, or $z = F^{-1}F^t x$, since F is non-degenerate.

Lemma 3.1. If P is an (l, m) self-dual n-gon, then $G(A_i) = A_{i+2l+m}$.

Proof. We know that $A_i^{\perp} = B_{i+l}^m$, and therefore

$$(A_i^{\perp})^{\perp} = (B_{i+l}^m)^{\perp} = B_{i+2l}^m \cap B_{i+2l+m}^m = A_{i+2l+m}.$$

□

It can also be seen that $G^r = \text{Id}$, where $r = \frac{n}{(n, 2l+m)}$ since

$$G^r(A_i) = A_{i+r(2l+m)} = A_i$$

for all i, and a projective isomorphism that fixes four points in $\mathbb{P}_2$ is the identity.

Lemma 3.2. In appropriate coordinates, $F = H_\phi$ where $\phi = \frac{k}{r}\pi$ and $1 \le k < 2r$, such that $(k,r) = 1$

Proof. Looking back at Theorem 2.14 we notice that

$$J^{-1}J^t = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K^{-1}K^t = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$$

have infinite orders and therefore cannot equal G. Therefore, $F = H_\phi$. Also, $G = F^{-1}F^t = H_{-2\phi}$, and since $G^r = \mathrm{Id}$, $2r\phi$ is a multiple of 2π. The condition $(k,r) = 1$ ensures that $G^s \ne \mathrm{Id}$ for all $s < r$ which is necessary since the polygon is simple. ☐

This goes to say that an (l,m) self-dual n-gon, where $2l+m \ne n$ consists of $(n, 2l+m)$ regular $\frac{n}{(n,2l+m)}$-gons.

3.3.1 Explicit Construction.

Given n, l, m with the necessary relations, we are going to construct all (l,m) self-dual n-gons. Let $r = \frac{n}{(n,2l+m)}$, and fix k such that $1 \le k < 2r$ and $(k,r) = 1$. Let $\phi = \frac{k}{r}\pi$. Fix a basis and let $F = H_\phi$ and consequently $G = H_{-2\phi}$. There are two different cases to investigate based on the combinatorics of n, l and m.

Case 1: There exists $a \in \mathbb{N}$ such that $a(2l+m) \cong l \mod n$.

> **Proposition 3.3.** Let P be an n-gon, and let l,m be fixed. Let a basis be fixed and $F = H_\phi$, where $\phi = \frac{k}{r}\pi$, $1 \le k < 2r$ and $(k,r) = 1$. Let $G = F^{-2}$ and suppose that $A_{i+2l+m} = GA_i$ for all i and that $a(2l+m) \cong l \mod n$. Then P is (l,m) self-dual with respect to F if and only if $A_i^t F^{1-2a} A_i = 0$ for all i.

Proof. We need to show that this condition is equivalent to

$$A_i^t F A_{i+l} = A_i^t F A_{i+l+m} = 0.$$

Since $A_{i+l} = G^a A_i = F^{-2a} A_i$,

$$A_i^t F A_{i+l} = A_i^t F(F^{-2a} A_i) = A_i^t F^{1-2a} A_i.$$

Also notice that $(1-a)(2l+m) \cong l+m \mod n$. Therefore

$$A_{i+l+m} = G^{1-a} A_i = F^{-2+2a} A_i$$

and

$$A_i^t F A_{i+l+m} = A_i^t F(F^{-2+2a} A_i) = A_i^t F^{-1+2a} A_i = A_i^t F^{1-2a} A_i.$$

The last equality is obtained by taking the transpose of the entire expression and noting that $F^t = F^{-1}$. □

Therefore, to construct an (l,m) self-dual n-gon in case 1, we need to pick A_0 on the affine plane defined by the chart U_{x_2} such that $A_0^t F^{1-2a} A_0 = 0$.[3] This restricts A_0 to the conic defined by $x^t F^{1-2a} x = 0$. Consequently the points $\{A_{2l+m}, A_{2(2l+m)}, \dots, A_{(r-1)(2l+m)}\}$ are fixed by the relation $GA_i = A_{i+2l+m}$. It is an easy exercise to show that $G^s A_0$ also lies on this conic for all s. If this exhausts all points, then we are done. If not, we pick A_1 on the same conic and continue the same process. In the end we will have made the choice of a point on the conic $(n, 2l+m)$ times. This is the dimension of the set of (l,m) self-dual n-gons in this case once the basis is fixed.
Also note that this polygon is inscribed into a conic defined by $xF^{1-2a}x = 0$.

[3]If we pick A_0 on the line at infinity then A_m and A_{2m} will also lie on the line at infinity since images of points at infinity under H_ϕ are also at infinity. This contradicts the non-degeneracy of the n-gon P.

Case 2: $a(2l+m) \ncong l$ for all $a \in \mathbb{N}$.

Let a be such that $al \cong b(2l+m) \mod n$. To construct the (l,m) self-dual polygon in this case, pick $A_0 \neq 0$ anywhere in the affine plane. All points of the form $A_{c(2l+m)}$ are fixed by the relation $A_{i+2l+m} = F^{-2}A_i$. Pick $A_l \in A_0^{\perp}$ and continue picking $A_{cl} \in A_{(c-1)l}^{\perp}$ until you reach $A_{(a-1)l}$ which you fix as $A_{(a-1)l} = A_{(a-2)l}^{\perp} \cap A_{(b-1)(2l+m)}^{\perp}$. The points of the form $A_{dl+c(2l+m)}$ are again fixed by the relation $A_{i+2l+m} = F^{-2}A_i$. If this takes care of all the points, we are done. If not, then pick $A_1 \neq 0$ anywhere in the affine plane and repeat the same procedure until we have chosen all points.

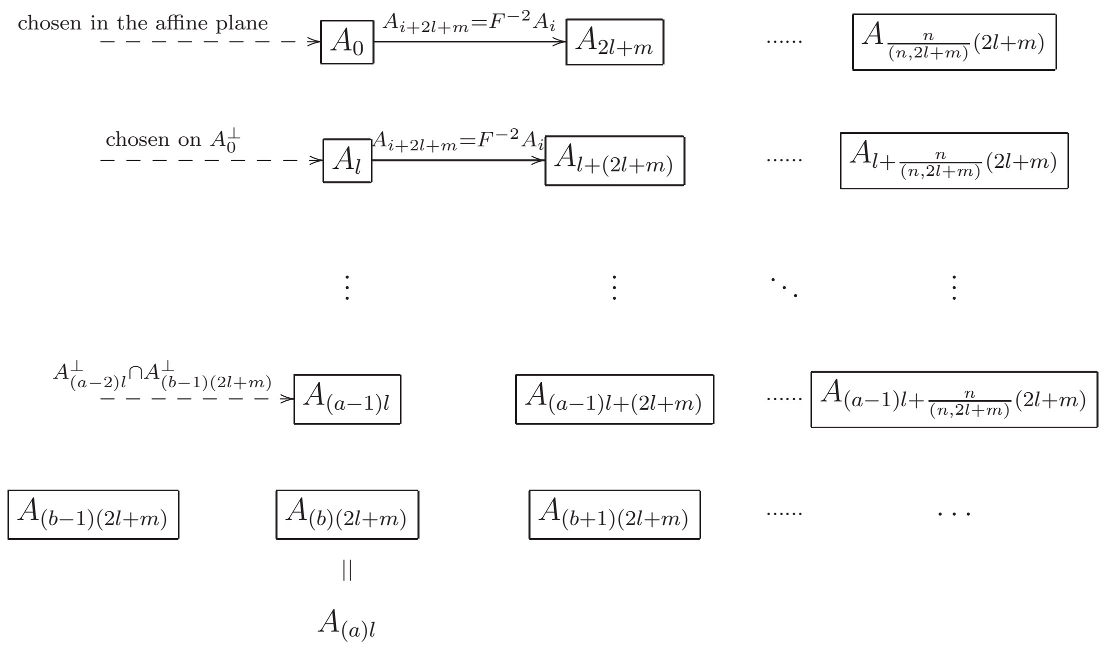

Figure 10: Scheme for the construction (l,m) of self dual polygons.

Proposition 3.4. Under this construction, $A_{i+l} \in A_i^{\perp}$ for all i.

Proof. This is obvious for almost all points except $A_{al} = A_{b(2l+m)}$. Notice that

$A_{b(2l+m)} = F^{-2}A_{(b-1)(2l+m)}$ and since $A_{(a-1)l} \in A^{\perp}_{(b-1)(2l+m)}$

$$\begin{aligned} 0 &= A^t_{(b-1)(2l+m)} F A_{(a-1)l} = (F^2 A_{b(2l+m)})^t F A_{(a-1)l} \\ &= A^t_{(a-1)l} F A_{b(2l+m)} \end{aligned}$$

Which exactly implies that $A_{al} \in A^{\perp}_{(1-a)l}$. □

Proposition 3.5. Under this construction, $A_{i+l+m} \in A^{\perp}_i$ for all i.

Proof. Since $A_i \in A^{\perp}_{i-l}$ and $A_{i-l} = F^2 A_{i+l+m}$,

$$\begin{aligned} 0 &= A^t_{i-l} F A_i = (F^2 A_{i+l+m})^t F A_i \\ &= A^t_i F A_{i+l+m}. \end{aligned}$$

□

This shows that our polygon is (l, m) self-dual, and it is clear why this construction takes care of all (l, m) self-dual polygons since every restriction was a necessity for the polygon to be (l, m) self-dual. Notice also that if $2l \cong b(2l + m) \mod n$, then we do not choose A_l on $A^{\perp}_0$ but fix it as $A_l = A^{\perp}_0 \cap A^{\perp}_{-m}$.

We can now deduce that the dimension of the set of (l, m) self-dual polygons is again $(n, 2l + m)$ once the basis is fixed. In Diagram 10, each row represents an equivalence class of points with indices differing by a multiple of $(2l + m)$. During the algorithm, we encounter the same number of degrees of freedom as the number of equivalence classes that we define. For the first equivalence class we choose A_0 anywhere on the plane, for all other equivalence classes we pick a point on a line except for the last equivalence class which is fixed by previous choices. The total number of these equivalence classes is $(n, 2l + m)$.

Proposition 3.6. The dimension of the moduli space of all (l, m) self-dual polygons where $2l + m \neq n$ is $(n, 2l + m) - 3$ for $n = 2(2l + m)$, and $(n, 2l + m) - 1$ for $n \neq 2(2l + m)$.

Proof. We have shown that if the basis is fixed then the dimension of the set of (l, m) self-dual polygons is $(n, 2l+m)$. From this we have to subtract the dimension of the group of transformations that preserve F. From the proof of Theorem 2.14, this dimension is 1 if 2ϕ is not a multiple of π and 3 if 2ϕ is a multiple of π. Since $\phi = \frac{k}{r}\pi$ where $(k, r) = 1$, 2ϕ is a multiple of π if and only if $r = \frac{n}{(n, 2l+m)} = 2$, or equivalently $2(2l+m) = n$ since $2l+m < n$. ☐

Example 3.7. Let $n = 12$, $l = m = 1$. Then $2l + m = 3 \neq 12 = n$. We are going to construct a (1,1) self-dual 12-gon. We set $r = \frac{12}{(12,3)} = 4$, and pick a k s.t. $(k, r) = 1$. Let $k = 3$. Then we have

$$\phi = \frac{3}{4}\pi, \quad F = H_{\frac{3}{4}\pi}, \quad G = H_{-\frac{3}{2}\pi} = H_{\frac{1}{2}\pi}.$$

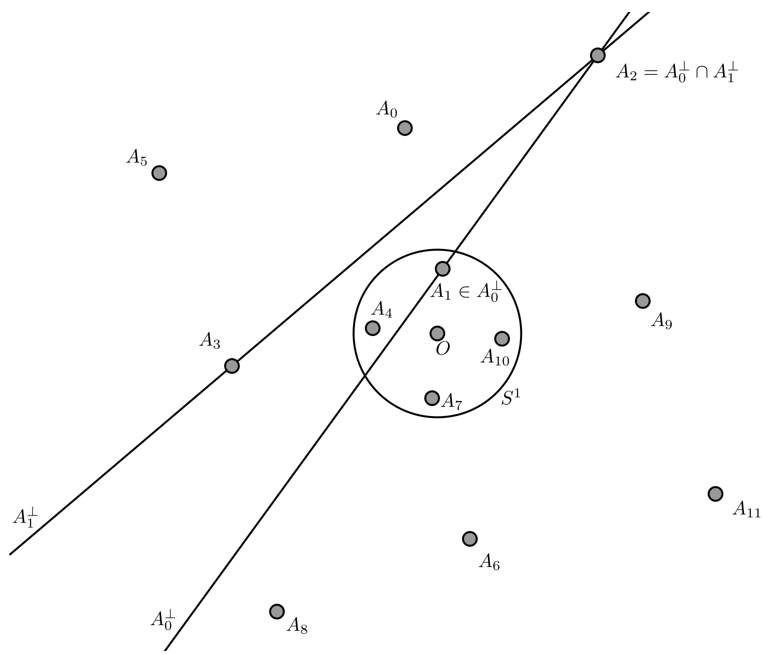

Figure 11: A $(1, 1)$ self-dual 12-gon.

Therefore G is a rotation by $\frac{1}{2}\pi$ in the counter-clockwise direction. Now we need to figure out how the line $A_0^\perp$ relates to A_0. For this purpose notice this

$$A_0^\perp = \{x \in \mathbb{P} | A_0^t H_\phi x = x^t H_{-\phi} A_0 = 0\}.$$

Therefore $A_0^\perp$ is the polar dual of $H_{-\phi}A_0$. Since in our case $\phi = \frac{3}{4}\pi$, in order to get $A_0^\perp$ we need to rotate A_0 by $\frac{3}{4}\pi$ in the clockwise direction, then reflect the point in the unit circle, then reflect it in the origin and then draw a line through the resulting point perpendicular to its position vector.

Fix the point A_0 in the plane. Pick a point A_1 on $A_0^\perp$ and fix $A_2 = A_0^\perp \cap A_1^\perp$. The rest of the points are fixed by the relation

$$A_{i+2l+m} = A_{i+3} = H_{\frac{1}{2}\pi}A_0.$$

This gives us an $(1,1)$ self-dual 12-gon as in the Figure 11.

4 Geometric Surprises

This thesis was partially motivated by recently discovered relations between n-gons which are inscribed into a projective conic and those having the self-duality property discussed in Section 3. These relations have originally been discovered via computer experimentation. One can read more about these relations in [9].

Let χ_n and χ_n^* be the sets of n-gons in $\mathbb{P}_2$ and $\mathbb{P}_2^*$ respectively. We define a map $T_k : \chi_n \to \chi_n^*$ in the following way. Let $P = \{A_0, A_1, \ldots, A_{n-1}\}$ be a polygon and let $B_i^{m*} = [A_i A_{i+m}]^*$ be the duals of the m-diagonals of P. Then

$$T_m(P) = \{B_0^{m*}, B_1^{m*}, \ldots, B_{n-1}^{m}{}^*\}.$$

We will also use an abbreviation, $T_{ab} = T_a \circ T_b$. This makes sense, since the map T_m is also well defined on χ_n^*.

Given two n-gons $P \in \mathbb{P}(V)$ and $Q \in \mathbb{P}(U)$, we will say that P and Q are equivalent, or $P \sim Q$, if there exists a projective isomorphism $f : \mathbb{P}(V) \to \mathbb{P}(U)$ which sends P to Q preserving the order of vertices, but perhaps rotating them.

Proposition 4.1. The map T_m is an involution up to rotation of vertices and projective isomorphisms. In other words

$$T_{mm}(P) \sim P$$

for all n-gons P.

Proof. Let $P = \{A_0, A_1, \ldots, A_{n-1}\}$. Then $T_m(P) = \{B_0^{m*}, B_1^{m*}, \ldots, B_{n-1}^{m}{}^*\}$ and

$$\begin{aligned} T_{mm}(P) &= \{[B_0^{m*}, B_m^{m*}], [B_1^{m*}, B_{1+m}^{m}{}^*], \ldots, [B_{n-1}^{m}{}^*, B_{m-1}^{m}{}^*]\} \\ &= \{A_m, A_{m+1}, \ldots, A_{m-1}\} \end{aligned}$$

□

Definition 11. A non-singular projective conic C in $\mathbb{P}(V)$ is given by the equation

$$F(v, v) = 0$$

for some non-degenerate symmetric bilinear form F on V.

By Proposition 2.12, we know that any two non-singular conics are projectively equivalent. A polygon $P = \{A_0, A_1, \ldots, A_{n-1}\}$ will be called inscribed if there exists a projective conic C such that $A_i \in C$ for all i. We will also call a polygon P, m-self-dual, if it is (l, m) self-dual for some l.

We are now able to state the theorem which should astound the reader.

Theorem 4.2. Let $P \subset \mathbb{CP}_2$ be an inscribed n-gon. Then the following statements hold

If P is a 6-gon then $P \sim T_2(P)$

If P is a 7-gon then $P \sim T_{212}(P)$

If P is an 8-gon then $P \sim T_{21212}(P)$

If P is a 9-gon then $P \sim T_{13131}(P)$

If P is a 12-gon then $P \sim T_{3434343}(P)$

An equivalent formulation of Theorem 4.2 is that if P is an inscribed n-gon then the following statements hold

If P is a 6-gon then P is 2-self-dual

If P is a 7-gon then $T_2(P)$ is 1-self-dual

If P is an 8-gon then $T_{12}(P)$ is 2-self-dual

If P is a 9-gon then $T_{31}(P)$ is 1-self-dual

If P is a 12-gon then $T_{343}(P)$ is 4-self-dual

The reason why the two collections of statements are equivalent is that T_m is an involution. One might notice some patterns in the theorem and wonder if they are beginnings of some infinite patterns. As far as computer simulations go, no similar patterns were noticed for n-gons where n is greater than 12.

Although Theorem 4.2 has a purely geometrical flavor, only the first two statements gave in to geometrical arguments. The rest of the theorems had to be proved computationally. These proofs are not very insightful, and one should still ask oneself whether there is some underlying reasons why this theorem holds and why there are no similar statements for n-gons where n is greater than 12. In this thesis we will present the geometrical proofs.

4.1 Corner Invariants

In order to analyze polygons up to projective isomorphisms, we shall introduce coordinates on the space of polygons which are preserved under projective transformations.

Definition 12. Let $P = \{A_0, A_1, \ldots, A_{n-1}\}$ be an n-gon. Consider the following construction at a point A_i. Let $P = [A_{i-2}, A_{i-1}] \cap [A_i, A_{i+1}]$, $R = [A_{i-1}, A_{i-2}] \cap [A_{i+1}, A_{i+2}]$, and $Q = [A_{i-1}, A_i] \cap [A_{i+1}, A_{i+2}]$ as in Figure 12. Then the corner invariants at the point A_i are defined as

$$p_i = [A_{i-2}, A_{i-2}, P, R] \qquad\qquad q_i = [R, Q, A_{i+1}, A_{i+2}]$$

where $[*, *, *, *]$ stands for cross-ratio.

It is clear that corner invariants stay invariant under projective transformations since cross-ratio is invariant under projective transformations. Also, corner invariants define the n-gon uniquely. By Proposition 2.4, the first four points can be fixed anywhere in the plane. All the following points can be constructed afterward using corner invariants. The corner invariants are not independent. They have to satisfy certain relations in order for the polygon to be closed.

It turns out that the map T_1 behaves nicely with respect to corner invariants. This will be important for us in one of the geometric proofs.

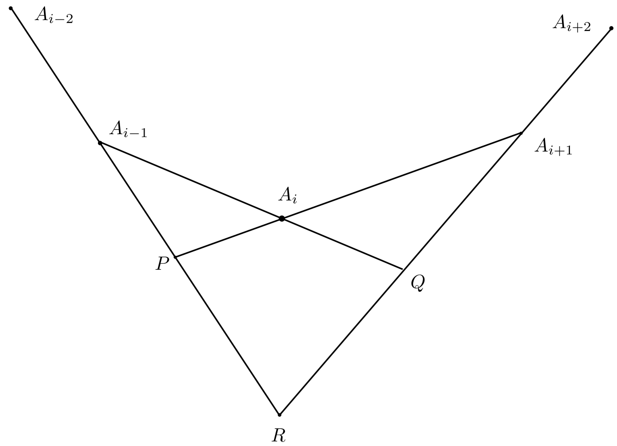

Figure 12: Corner invariants at a point A_i.

Proposition 4.3. Let $P = \{A_0, A_1, \ldots, A_{n-1}\}$ be an n-gon. Let p_i, q_i be the corner invariants of P and let p_i^*, q_i^* be the corner invariants of $T_1(P)$. Then

$$p_i^* = q_i \qquad\qquad q_i^* = p_{i+1}$$

Proof. We will only show the first relation. The second relation is obtained by applying the first relation to the polygon $T_1(P)$ and realizing that the polygon $T_{11}(P)$ is the rotation of P by one vertex.

To calculate p_i^* we need to construct $P^* = [{B_{i-2}^1}^*, {B_{i-1}^1}^*] \cap [B_i^{1*}, {B_{i+1}^1}^*]$ and $R^* = [{B_{i-1}^1}^*, {B_{i-2}^1}^*] \cap [{B_{i+1}^1}^*, {B_{i+2}^1}^*]$ as in Figure 13. In the dual space, $[{B_{i-2}^1}^*, {B_{i-1}^1}^*]$ is the line passing through points ${B_{i-2}^1}^*$ and ${B_{i-1}^1}^*$ which in the original space takes the form of a point of intersection of B_{i-2}^1 and B_{i-1}^1. Also the intersection of two lines l_1, l_2 in the dual space takes the form of a line passing through l_1^* and l_2^* in the original space.

By Proposition 2.10, the cross-ratio of the four lines equals the cross ratio of four points which are intersections of those four lines with some fixed line. Therefore

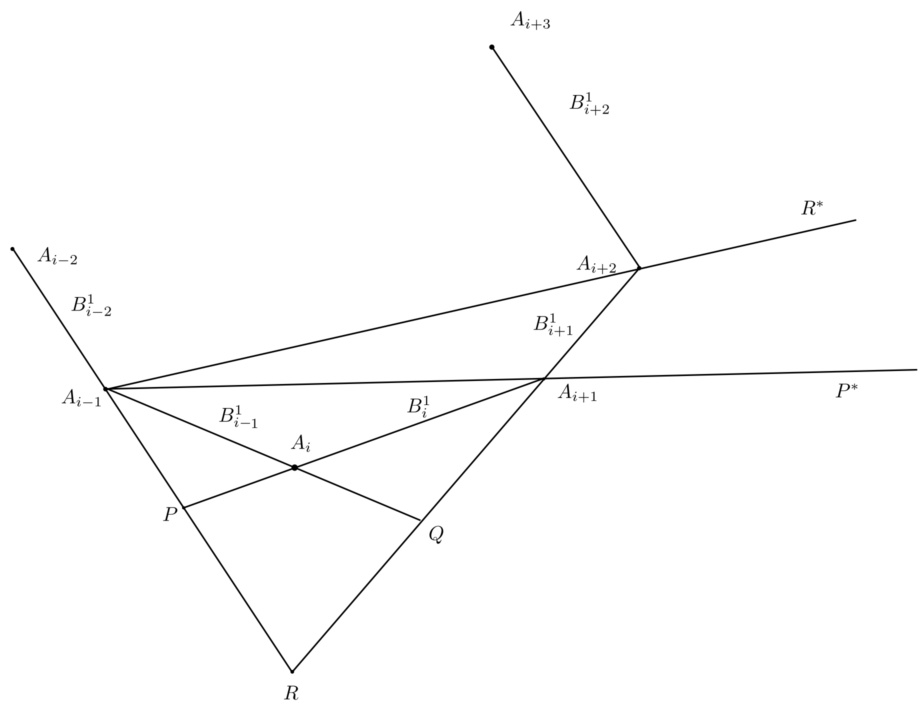

Figure 13: Corner invariants of the dual polygon.

projecting lines $B^1_{i-2}, B^1_{i-1}, P^*, Q^*$ onto the line $[A_{i+1}, A_{i+2}]$ gives us

$$p^*_i = [{B^1_{i-2}}^*, {B^1_{i-1}}^*, P^*, Q^*] = [R, Q, A_{i+1}, A_{i+2}] = q_i.$$

□

4.2 Geometrical Proofs

At the heart of the geometrical proofs that will be presented in this section, lies a classical theorem that we will state without a proof.

Theorem 4.4 (Pascal's Theorem)**.** Let C be a conic. Let $A_1, A_2, A_3, B_1, B_2, B_3 \in C$

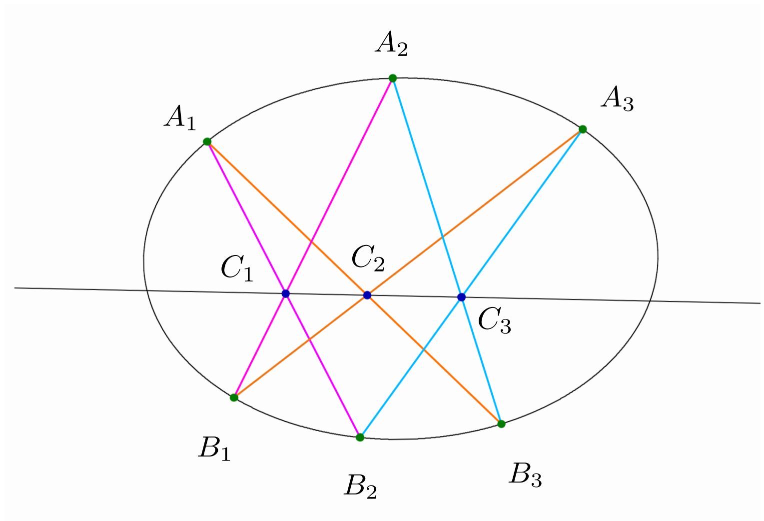

Figure 14: Pascal's Theorem Configuration.

be six points on the conic C. Let

$$C_1 = [A_1, B_2] \cap [A_2, B_1]$$
$$C_2 = [A_1, B_3] \cap [A_3, B_1]$$
$$C_3 = [A_2, B_3] \cap [A_3, B_2]$$

as in the Figure 14. Then the three points C_1, C_2, C_3 are collinear.

4.2.1 Inscribed 6-gons.

The first statement which we will prove geometrically is that an inscribed 6-gon is 2-self dual.

Proof. Let $P = \{A_0, A_1, \ldots, A_5\}$ be an inscribed 6-gon. We will show that P is $(2, 2)$ self-dual. For that purpose we will show that corner invariants of $T_2(P)$ are the same as those of P but rotated by two. Let p_i, q_i be the corner invariants of P at the vertex A_i and let p_i^*, q_i^* be the corner invariants of $T_2(P)$ at the vertex B_i^{2*}. We will show that

$$p_0 = p_2^*.$$

The equality of the rest of the corner invariants follows from analogous reasonings.

The first thing that we need to do is find the eight points, whose cross-ratios determine p_0 and p_2^*. Let $P^* = [B_0^{1*}, B_1^{1*}] \cap [B_2^{1*}, B_3^{1*}]$, $R^* = [B_0^{1*}, B_1^{1*}] \cap [B_3^{1*}, B_4^{1*}]$ as in Figure 15. Then from the definition of corner invariants,

$$p_2^* = [B_0^{2*}, B_1^{2*}, P^*, R^*].$$

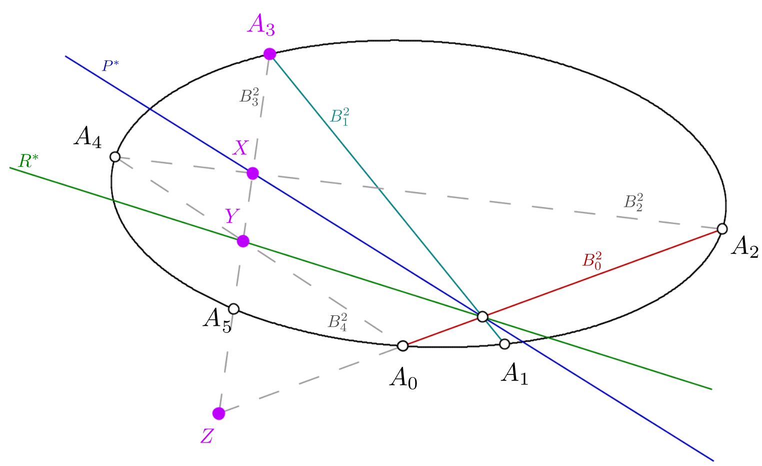

Figure 15: Calculation of p_2^*

If we let $Z = [A_3, A_5] \cap [A_0, A_2]$, $X = [A_4, A_2] \cap [A_3, A_5]$ and $Y = [A_0, A_4] \cap [A_3, A_5]$, and then project the four lines B_0^2, B_1^2, P^*, R^* onto the line $[A_3, A_5]$, we get that

$$p_2^* = [Z, A_3, X, Y].$$

Now we need to find p_0. If we let $P = [A_4, A_5] \cap [A_0, A_1]$, $R = [A_4, A_5] \cap [A_1, A_2]$, then by definition

$$p_0 = [A_4, A_5, P, R].$$

We need to show that $p_0 = p_2^*$ or equivalently

$$[A_4, A_5, P, R] = [Z, A_3, X, Y].$$

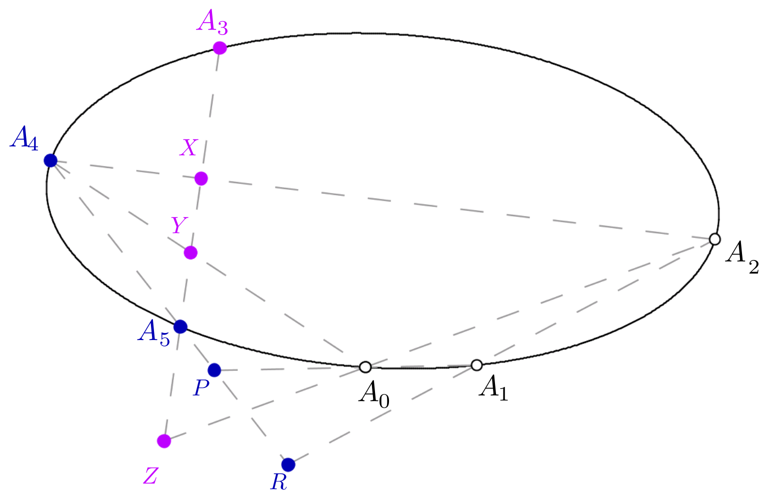

Figure 16: Calculation of p_0

The cross-ratio is preserved under projections. In order to show that the two cross-ratios are the same, we are going to construct a map ξ from the line $[A_3, A_5]$ to the line $[A_4, A_5]$ such that ξ is a composition of two projections and

$$\xi(Z) = R \qquad \xi(A_3) = P \qquad \xi(X) = A_5 \qquad \xi(Y) = A_4.$$

This will complete the proof since ξ leaves the cross-ratio invariant and also [X,Y,Z,W] = [W,Z,Y,X] for any quadruple of points $\{X, Y, Z, W\}$. (This can be directly verified from the definition of cross-ratio.)

Let $S = [A_3, A_5] \cap [A_1, A_2]$, and let $\xi = \pi_S \circ \pi_{A_0}$, where π_{A_0} is the projection from the line $[A_3, A_5]$ onto the line $[A_4, A_2]$ through the point A_0, and where π_S is the projection of the line $[A_4, A_2]$ onto the line $[A_4, A_5]$ through the point S.

We need to verify four images of the map ξ. Three of those images are obvious.

$$\xi(Z) = \pi_S \circ \pi_{A_0}(Z) = \pi_S(A_2) = R$$
$$\xi(X) = \pi_S \circ \pi_{A_0}(X) = \pi_S(X) = A_5$$
$$\xi(Y) = \pi_S \circ \pi_{A_0}(Y) = \pi_S(A_4) = A_4$$

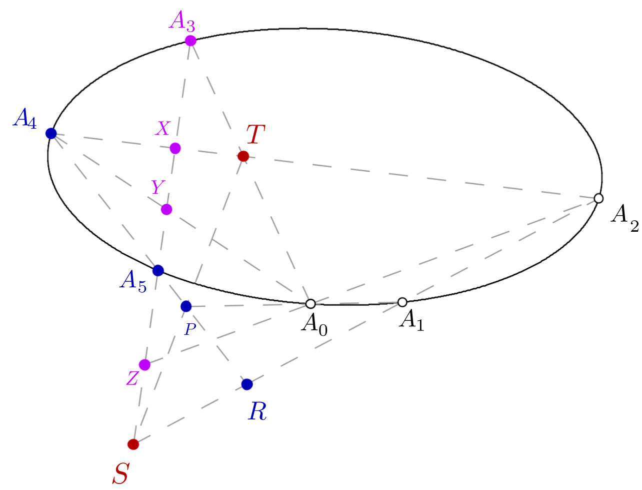

Figure 17: The map ξ

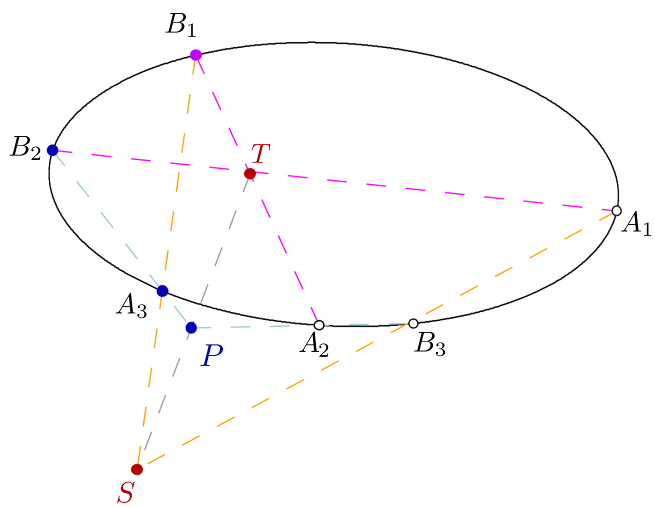

Figure 18: The map ξ and Pascal's Theorem Configuration

To verify the last image, let $\pi_{A_0}(A_3) = T$ as in Figure 17. Then $\xi(A_3) = \pi_S(T) = P$ if and only if the three points T, P, S are collinear. This is the moment when we need to apply Pascal's theorem. Drawing the same figure separately, and renaming the vertices of our polygon so that they correspond to Figure 14 of Pascals theorem, we get precisely the statement that we need, namely that T, P, S are collinear. $\square$

4.2.2 Inscribed 7-gons.

Now we will show that if P is an inscribed 7-gon, then $T_2(P)$ is 1-self-dual. This proof is due to Richard Schwartz.

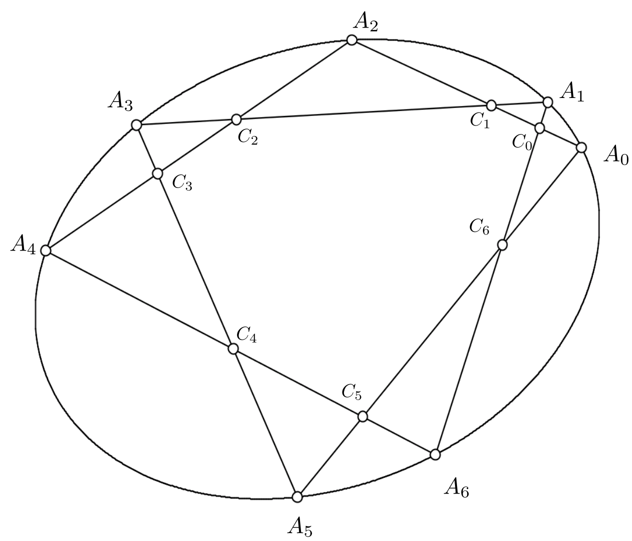

Figure 19: P and $T_{12}(P)$.

Proof. Let $P = \{A_0, A_1, \ldots, A_6\}$ be an inscribed 7-gon. We will show that $T_{12}(P)$ is $(3,1)$ self-dual. Clearly if T_{12} is $(3,1)$ self-dual, then $T_2(P) \sim T_{12}(P)$ and therefore T_2 is also $(3,1)$ self-dual. Let $T_{12}(P) = \{C_0, C_1, \ldots, C_6\}$. It is easy to verify that $C_i = [A_i, A_{i+2}] \cap [A_{i+1}, A_{i+3}]$. Let p_i, q_i be the corner invariants of $T_{12}(P)$ at the vertex C_i.

From Proposition 4.3, it follows that $T_{12}(P)$ is $(3,1)$ self-dual if and only if $p_i = q_{i+3}$ and $q_i = p_{i+4}$ for all i. We will show that $p_0 = q_3$. The other relations follow from the same logic.

The strategy for this proof is the same as for the one in the previous section. We will find the eight points which define p_0, q_3 and find a projection that sends one collection of four points to the other.

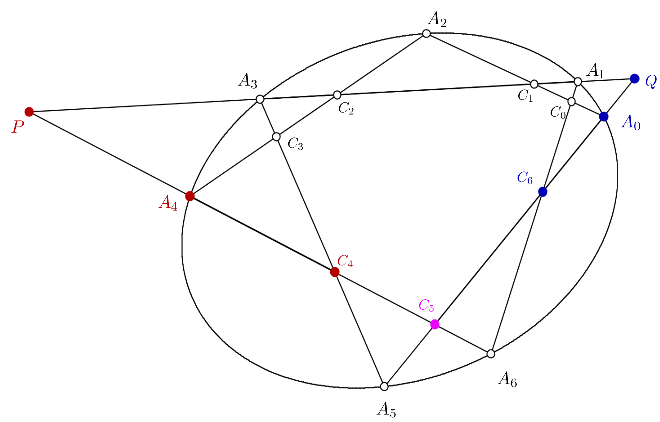

Figure 20: The Corner Invariants.

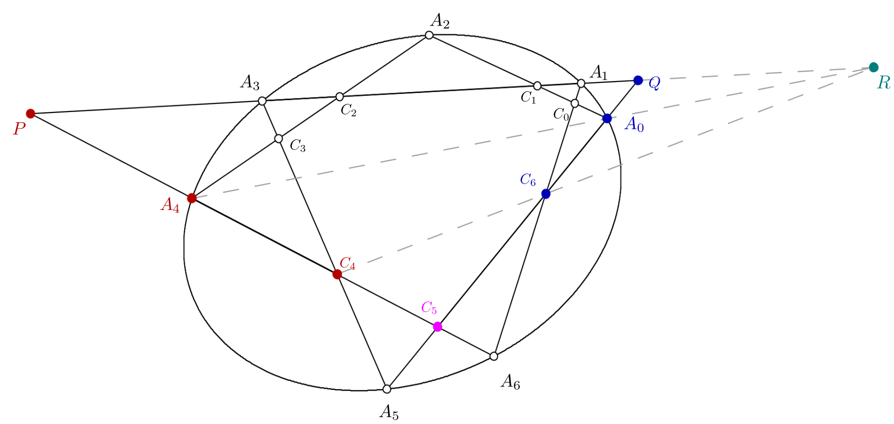

Figure 21: The Map π_R.

Let $P = [A_4, A_6] \cap [A_1, A_3]$ and $Q = [A_1, A_3] \cap [A_5, A_0]$ as in Figure 20. It is easy to check that

$$p_0 = [C_5, C_6, A_0, Q] \qquad\qquad q_3 = [C_5, C_4, A_4, P].$$

We need to find a point R such that π_R is a projection from the line $[A_0, A_5]$ onto the line $[A_4, A_6]$ such that

$$\pi_R(C_5) = C_5 \qquad \pi_R(C_6) = C_4 \qquad \pi_R(A_0) = A_4 \qquad \pi_R(Q) = P.$$

Let $R = [A_4, A_0] \cap [A_3, A_1]$. By Pascals theorem, R, C_6, C_4 are collinear. Taking a close look at Figure 21 one can easily see that π_R is the desired map and $p_0 = q_3$. $\square$

5 Conclusion

There is a computational proof for the statements of theorem 4.2 involving hexagons, heptagons and octagons due to Sergei Tabachnikov that requires one to calculate the action of the map T_2 in terms of corner invariants. The rest of the cases were proved by brute force computation using Mathematica and are due to Richard Schwartz and Sergei Tabachnikov. The paper with the details of these proofs should appear in the near future.

In this thesis we considered fixed points of the map T_m. A natural generalization of this analysis would be to consider the fixed points of the map T_{ab} for some natural numbers a, b. It turns out that the dynamics of the map T_{ab} are very rich. The following papers deal solely with the dynamics of the map T_{12} : [3–8]. Further, one can consider dynamics of the map $T_{a_1,a_2,\ldots,a_m}$ for some fixed sequence $a_1, a_2, \ldots, a_m$.

Another direction where one might take this research is to analyze the "geometric surprises". One might try to find a joint proof for all statements of Theorem 4.2 and find out why the pattern stops at 12-gons.

List of Figures

References

[1] M. Berger, *Geometry 1*, Springer, 1987.

[2] D. Fuchs and S. Tabachnikov, *Self-dual polygons and self-dual curves*, Funct. Anal. and Other Math. **2** (2009), 203–220.

[3] M. Glick, *The pentagram map and Y-patterns*, ArXiv e-prints (May 2010), available at `1005.0598`.

[4] V. Ovsienko, R. Schwartz, and S. Tabachnikov, *The Pentagram map: a discrete integrable system*, Comm. Math. Phys., in print.

[5] V. Ovsienko, R. Schwartz, and S. Tabachnikov, *Quasiperiodic motion for the pentagram map*, Electron. Res. Announc. Math. Sci. **16** (2009), 1–8.

[6] R. Schwartz, *The pentagram map*, Experiment. Math. **1** (1992), no. 1, 71–81.

[7] R. Schwartz, *The pentagram map is recurrent*, Experiment. Math. **10** (2001), no. 4, 519–528.

[8] R. Schwartz, *Discrete monodromy, pentagrams, and the method of condensation*, Journal of Fixed Point Theory and Applications **3** (2008), no. 2, 379–409.

[9] R. Schwartz and S. Tabachnikov, *Elementary surprises in projective geometry*, The Mathematical Intelligencer (2009).